

# WHITNEY THEOREM FOR COMPLEX POLYNOMIAL MAPPINGS

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ABSTRACT. We describe the topology of a generic polynomial mapping  $F = (f, g) : X \rightarrow \mathbb{C}^2$ , where  $X$  is the complex plane or a complex sphere.

## 1. INTRODUCTION

Polynomial mappings  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  are the most classical objects in the complex analysis, yet their topology has not been studied up till now. To the best knowledge of the authors complex algebraic families of polynomials on affine varieties have not been investigated so far. Here we describe an idea of such study. For a smooth affine variety  $X^k \subset \mathbb{C}^p$  we consider the family  $\Omega_X(d_1, \dots, d_m)$  of polynomial mappings  $F : X \rightarrow \mathbb{C}^m$  of degree bounded by  $(d_1, \dots, d_m)$ . In particular we prove a suitable version of Thom transversality theorem for this family, which is useful at least if  $d_i \geq k$ . We prove, that under this assumption a generic member of  $\Omega_X(d_1, \dots, d_m)$  is transversal to the Thom-Boardman strata in  $J^k(X, \mathbb{C}^m)$ .

Let us recall that in [10] the second author proved that if  $X, Y$  are smooth affine manifolds and  $\Phi : M \times X \rightarrow Y$  is an algebraic family of polynomial mappings, such that the generic element of this family is proper, then two generic members of this family are topologically equivalent. In particular if  $X \subset \mathbb{C}^p$  is of dimension  $n$  and  $m \geq n$  then any two generic members of the family  $\Omega_X(d_1, \dots, d_m)$  are topologically equivalent. For example, if  $X$  is a smooth surface, then the numbers  $c_X(d_1, d_2)$  and  $d_X(d_1, d_2)$  of cusps and double folds of a generic member of the family  $\Omega_X(d_1, d_2)$  are well-defined.

Our aim is to determine effectively the topology of such generic mappings. We consider in this paper the simplest case, when  $n = m = 2$  and  $X = \mathbb{C}^2$  or  $X$  is the complex sphere  $S = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = 1\}$ . In those cases we determine the topology of the set  $C(F)$  of critical points of  $F$  and the topology of its discriminant  $\Delta(F)$ . In particular we show that a generic polynomial mapping  $F \in \Omega_X(d_1, d_2)$  has only cusps, folds and double folds as singularities and we compute the number  $c_X(d_1, d_2)$  of cusps and number  $d_X(d_1, d_2)$  of double folds of such generic polynomial mapping. Our ideas work well also in higher dimensions. This paper is the first step in a study of the topology of generic polynomial mappings  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ .

The problem of counting the number of cusps of a generic perturbation of a real plane-to-plane singularity was considered by Fukuda and Ishikawa in [3]. They proved that the number modulo 2 of cusps of a generic perturbation  $F$  of a finitely determined map-germ  $F_0 : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  is a topological invariant of  $F_0$ . More recently, in [11] Krzyżanowska and Szafraniec gave an algorithm to compute the number of cusps for sufficiently generic fixed real polynomial mapping of the real plane.

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Algebraic formulas to count the number of cusps and nodes of a generic perturbation of a finitely determined holomorphic map-germ  $F_0 : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ , were given by Gaffney and Mond in [4, 5] (see also [16]). In this case any two generic perturbations  $F$  of  $F_0$  defined on a sufficiently small neighborhood of 0 are topologically equivalent, so the number of cusps and nodes of  $F$  is an invariant of the map-germ  $F_0$ .

Let us note that in some cases our result allows also to use local methods to study global mappings. Indeed, in the special case when  $\gcd(d_1, d_2) = 1$  the numbers  $c(F)$  and  $d(F)$  can be computed by using local methods of Gaffney and Mond [5] and J. J. Nuno-Ballesteros, B. Orefice-Okamoto, J. N. Tomazella [14], or Ohmoto methods [15] based on Thom polynomials. Note that in this case the leading homogenous part  $F_h$  of a generic mapping  $F = (f, g)$  is finitely determined. Moreover, we have a deformation  $F_t(x) = (t^{d_1} f(t^{-1}(x)), t^{d_2} g(t^{-1}(x)))$ . Now we can use the fact (which is first proved in our paper) that a generic (with respect to the Zariski topology) mapping  $F \in \Omega_X(d_1, d_2)$  has only folds, cusps and double folds as singularities. Thus for the deformation  $F_t \in \Omega_X(d_1, d_2)$  of  $F$  all  $F_t$ ,  $t \neq 0$  are generic mappings and all cusps and nodes of  $F_t$  tend to 0 when  $t \rightarrow 0$ . In this case our formulas for  $c(F)$  and  $d(F)$  coincide with formulas of Gaffney-Mond etc.

However, in the general case these approaches do not work since any homogeneous mapping is not finitely determined if  $\gcd(d_1, d_2) \neq 1$  (Gaffney-Mond, [5]). In particular in that case the local number of nodes can not be defined and the methods of Gaffney-Mond, Nuno-Ballesteros-Orefice-Okamoto-Tomazella and Ohmoto do not work. If  $\gcd(d_1, d_2) \neq 1$  our formulas do not coincide with formulas of Gaffney-Mond, Nuno-Ballesteros-Orefice-Okamoto-Tomazella and Ohmoto. Hence in general even discrete global invariants can not be obtained by local methods or methods based on Thom polynomials.

Now we will briefly describe the content of the paper. In Section 2 we state and prove general theorems. In Section 3 we describe the topology of the set of critical points of a generic mapping  $F \in \Omega_{\mathbb{C}^2}(d_1, d_2)$ . Moreover we compute the number  $c_{\mathbb{C}^2}(d_1, d_2)$  of cusps. In Section 4 we describe the topology of the discriminant  $\Delta(F)$  and compute the number  $d_{\mathbb{C}^2}(d_1, d_2)$  of nodes of  $\Delta(F)$ . In Section 5 we describe the topology of the set of critical points of a generic mapping  $F \in \Omega_S(d_1, d_2)$ , and compute the number  $c_S(d_1, d_2)$ , where  $S \subset \mathbb{C}^3$  is a complex sphere. In section 6 we describe the topology of the discriminant  $\Delta(F)$  and we compute the number  $d_S(d_1, d_2)$ .

We conclude the paper with Section 7 where we introduce the notions of a generalized cusp and the index of a generalized cusp  $\mu$  (see Definitions 7.1 and 7.3). We show that if  $F = (f, g) : X \rightarrow \mathbb{C}^2$  is an arbitrary polynomial mapping with  $\deg f \leq d_1$ ,  $\deg g \leq d_2$  and generalized cusps at points  $a_1, \dots, a_r$  then  $\sum_{i=1}^r \mu_{a_i} \leq c_X(d_1, d_2)$ .

## 2. GENERAL POLYNOMIAL MAPPINGS

Let  $\Omega_n(d_1, \dots, d_m)$  denote the space of polynomial mappings  $F : \mathbb{C}^n \rightarrow \mathbb{C}^m$  of multi-degree bounded by  $d_1, \dots, d_m$ . Similarly if  $X \subset \mathbb{C}^p$  is a smooth affine variety we consider a family  $\Omega_X(d_1, \dots, d_m)$  of polynomial mappings  $F : X \rightarrow \mathbb{C}^m$  of multi-degree bounded by  $d_1, \dots, d_m$  (in the sense that  $F \in \Omega_X(d_1, \dots, d_m)$  if there exists  $\tilde{F} \in \Omega_p(d_1, \dots, d_m)$  such that  $F = \tilde{F}|_X$ , note that for a generic  $F \in \Omega_p(d_1, \dots, d_m)$  the multi-degrees of  $F$  and  $F|_X$  coincide). Of course  $\Omega_X(d_1, \dots, d_m)$  has the structure of the affine space.

By  $J^q(\mathbb{C}^n, \mathbb{C}^m)$  we denote the space of  $q$ -jets of polynomial mappings  $F = (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ . We define it exactly as in [8]. However, if we fix coordinates in the domain and the target then we can identify  $J^q(\mathbb{C}^n, \mathbb{C}^m)$  with the space  $\mathbb{C}^n \times \mathbb{C}^m \times (\mathbb{C}^{N_q})^m$ , where  $\mathbb{C}^{N_q}$  parameterizes coefficients of polynomials of  $n$ -variables and of degree bounded by  $q$  with zero constant term (which correspond to suitable Taylor polynomials). In further

applications, in most cases, we treat the space  $J^q(\mathbb{C}^n, \mathbb{C}^m)$  in this simple way. In particular for a given polynomial mapping  $F : \mathbb{C}^n \rightarrow \mathbb{C}^m$  we can define the mapping  $j^q(F)$  as

$$j^q(F) : \mathbb{C}^n \ni x \mapsto \left( x, F(x), \left( \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) \right)_{1 \leq i \leq n, 1 \leq |\alpha| \leq q} \right) \in J^q(\mathbb{C}^n, \mathbb{C}^m).$$

If  $X^n \subset \mathbb{C}^p$  is a smooth affine variety, then the space  $J^q(X, \mathbb{C}^m)$  has the structure of a smooth algebraic manifold and can be locally represented in the same simple way as above. Indeed, locally  $X$  is a complete intersection, i.e. for every point  $x \in X$  there is an open neighborhood  $U_x$  of  $x$  such that  $U_x = \{g_1 = 0, \dots, g_{p-n} = 0\}$  (in some open set of  $\mathbb{C}^p$ ) and  $\text{rank}[\frac{\partial g_i}{\partial x_j}] = p - n$  on  $U_x$ . We can assume that the mapping  $(x_1, \dots, x_n, g_1, \dots, g_{p-n})$  is biholomorphic near  $x$ . In particular we have  $x_i = \phi_i(x_1, \dots, x_n)$  for  $i > n$ . Hence there exists another Zariski open neighborhood  $V_x$  of  $x$  such that in  $V_x$  we have global local coordinates  $x_1, \dots, x_n$ . In particular  $J^q(V_x, \mathbb{C}^m)$  can be identified with the space  $V_x \times \mathbb{C}^m \times (\mathbb{C}^{N_q})^m$ , where  $\mathbb{C}^{N_q}$  parameterizes coefficients of polynomials of  $n$ -variables and of degree bounded by  $q$  with zero constant term (which correspond to suitable Taylor polynomials). In local coordinates we have a mapping

$$j^q(F) : V_x \ni x \mapsto \left( x, F(x), \left( \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) \right)_{1 \leq i \leq n, 1 \leq |\alpha| \leq q} \right) \in J^q(V_x, \mathbb{C}^m).$$

We start with the following fact:

**Theorem 2.1.** *Let  $X^n \subset \mathbb{C}^p$  be a smooth affine variety of dimension  $n$ . Let  $S_1, \dots, S_k$  be smooth algebraic submanifolds of  $J^q(X, \mathbb{C}^m)$ . Let  $d_1, \dots, d_m$  be integers such that  $d_i \geq q$  for  $i = 1, \dots, m$ . Then there is a Zariski open dense subset  $V(S_1, \dots, S_k) \subset \Omega_X(d_1, \dots, d_m)$  such that for every  $F \in V(S_1, \dots, S_k)$  we have*

$$j^q(F) \pitchfork S_i, \text{ for } i = 1, \dots, k.$$

*Proof.* First consider the case  $X = \mathbb{C}^n$ ,  $p = n$ . For simplicity we can take  $m = 1$  (the general case is analogous). Consider the mapping

$$\Psi : \Omega_n(d_1) \times \mathbb{C}^n \ni (F, x) \mapsto j^q(F)(x) \in J^q(X, \mathbb{C}).$$

we have

$$\Psi = (\Psi_{(-1,0,\dots,0)}, \dots, \Psi_{(0,\dots,0,-1)}, \Psi_{(0,0,\dots,0)}, (\Psi_\alpha)_{1 \leq |\alpha| \leq q}) = (\Psi_\alpha, )_{-1 \leq |\alpha| \leq q},$$

where  $\Psi_{(-1,0,\dots,0)}(F, x) = x_1$ , etc. and  $\Psi_\alpha(F, x) = \frac{\partial^{|\alpha|} F}{\partial x^\alpha}(x)$ . Let  $a_{(0,\dots,-1,0,\dots,0)} := x_i$  (here  $-1$  is on  $i^{\text{th}}$ -position) be the coordinates in  $\mathbb{C}^n$  and  $a_\alpha$  for  $|\alpha| \geq 0$  be the coordinates in  $\Omega_n(d_1)$  such that  $F(x) = \sum a_\alpha x^\alpha$ .

We compute the matrix  $\left[ \frac{\partial \Psi_\alpha}{\partial a_\alpha} \right]_{-1 \leq |\alpha| \leq q}$ . It is easy to see that

$$\det \left[ \frac{\partial \Psi_\alpha}{\partial a_\alpha} \right]_{-1 \leq |\alpha| \leq q} = \prod_{q \geq |\alpha| > 0} (\alpha!) \neq 0.$$

Hence  $\Psi$  is a submersion.

Now assume that  $X$  is a general affine smooth variety. As above we can cover  $X$  by finite number of Zariski open subsets  $U_i$  which have global local coordinates  $x_{i_1}, \dots, x_{i_n}$ .

Let  $\Omega_n(d_1, \dots, d_m)(x_{i_1}, \dots, x_{i_n}) \subset \Omega_X(d_1, \dots, d_m)$  denote the set of polynomial mappings, which depend only on variables  $x_{i_1}, \dots, x_{i_n}$ . Note that we have

$$\Omega_X(d_1, \dots, d_m) \cong \Omega_n(d_1, \dots, d_m)(x_{i_1}, \dots, x_{i_n}) \oplus W,$$

where mappings in  $W$  have parameters independent from parameters in  $\Omega_n(d_1, \dots, d_m)(x_{i_1}, \dots, x_{i_n})$ . If we restrict our attention to the set  $U_i$  we have

$$\Omega_X(d_1, \dots, d_m) \cong \Omega_n(d_1, \dots, d_m)(x_{i_1}, \dots, x_{i_n}) \oplus W_i,$$

where  $W_i$  denotes the set of holomorphic mappings of variables  $x_{i_1}, \dots, x_{i_n}$ , which are defined on  $U_i$  and which have parameters independent from parameters in  $\Omega_n(d_1, \dots, d_m)(x_{i_1}, \dots, x_{i_n})$ . Now we can prove as above that  $\Psi : \Omega_X(d_1, \dots, d_m) \times U_i \ni (F, x) \mapsto j^q(F)(x) \in J^q(U_i, \mathbb{C}^m)$  is a submersion (in the proof it is enough to use only parameters from  $\Omega_n(d_1, \dots, d_m)(x_{i_1}, \dots, x_{i_n})$ ).

Fix  $1 \leq i \leq k$ . By the transversality theorem with a parameter the set of polynomials  $F \in \Omega_X(d_1, \dots, d_n)$  such that  $j^q(F)$  is transversal to  $S_i$  is dense in  $\Omega_X(d_1, \dots, d_m)$ . On the other hand this set is constructible in  $\Omega_X(d_1, \dots, d_m)$ .

We conclude that there is a Zariski open dense subset  $V_i \subset \Omega_X(d_1, \dots, d_m)$  such that for every  $F \in V_i$  we have  $j^q(F) \pitchfork S_i$ . Now it is enough to take  $V(S_1, \dots, S_k) = \bigcap_{i=1}^k V_i$ .  $\square$

**Definition 2.2.** Let  $S_k \subset J^1(X, \mathbb{C}^n)$  denote the subvariety of 1-jets of corank  $k$ . Let  $F \in \Omega_X(d_1, \dots, d_n)$ . We say that  $F$  is one-generic if  $F$  is proper and  $j^1(F) \pitchfork S_1$ .

By Theorem 2.1 the subset of one-generic mappings contains a Zariski open dense subset of  $\Omega_X(d_1, \dots, d_n)$ . We have the following result:

**Theorem 2.3.** *Let  $X$  be a smooth Stein manifold of dimension  $n$ . Let  $F : X \rightarrow \mathbb{C}^n$  be a proper holomorphic one-generic mapping. Let  $C(F)$  denote the set of critical points of  $F$ . Then there is an open and dense subset  $U \subset C(F)$  such that for every  $a \in U$  the germ  $F_a : (X_a, a) \rightarrow (\mathbb{C}^n, F(a))$  is holomorphically equivalent to a fold.*

*Proof.* Let  $\Delta = F(C(F))$  be the discriminant of  $F$ . Take  $U = C(F) \setminus F^{-1}(\text{Sing}(\Delta))$ . The set  $U$  is a Zariski open dense subset of  $C(F)$ . Take a point  $a \in U$  and consider the germ  $F_a : (X_a, a) \rightarrow (\mathbb{C}^n, F(a))$ . By the choice of the point  $a$  the germ of the discriminant of  $F_a$  is smooth. Hence by [9], Corollary 1.11, the germ  $F_a$  is biholomorphically equivalent to a  $k$ -fold:  $(\mathbb{C}^n, 0) \ni (x_1, \dots, x_n) \mapsto (x_1^k, x_2, \dots, x_n) \in (\mathbb{C}^n, 0)$ . In particular  $\text{corank}[F_a] = 1$ .

Now note that  $J^1(\mathbb{C}^n, \mathbb{C}^n) \cong \mathbb{C}^n \times \mathbb{C}^n \times M(n, n)$ , where  $M(n, n) = \{[a_{ij}], 1 \leq i, j \leq n\}$  is the set of  $n \times n$  matrices. In these coordinates the set  $S_1$  is given as  $\{(x, y, m) : \det[m_{ij}] = \phi(x, y, m) = 0\}$  on the open subset  $\{(x, y, m) : \text{corank}[m_{ij}] \leq 1\}$ . Since the mapping  $j^1(F)$  is transversal to  $S_1$  the mapping  $\phi \circ j^1(F) = kx_1^{k-1}$  has to be a submersion at 0. This is possible only for  $k = 2$ .  $\square$

In the sequel we use the Thom-Boardman singularities (see [1], [12]) which give a stratification in the jet space  $J^k(X, \mathbb{C}^m)$ . The strata are smooth and locally Zariski closed ([1], [12]). In fact we will use here mainly singularities of type  $S_i, S_{i,j}$  (see [1], [12] where they are denoted as  $\Sigma^i, \Sigma^{i,j}$ ).

We have the following general result:

**Theorem 2.4.** *Let  $X^k \subset \mathbb{C}^n$  be a smooth algebraic variety. Assume that  $d_i \geq k, i = 1, \dots, m$ . Then there is a Zariski open subset  $U \subset \Omega_X(d_1, \dots, d_m)$  such that for every  $F \in U$  the mapping  $F$  is transversal to the Thom-Boardman strata in  $J^k(X, \mathbb{C}^m)$ .*

*Proof.* Note that  $J^k(X, \mathbb{C}^m)$  is an algebraic variety and Thom-Boardman strata are smooth algebraic subvarieties. Now we can apply Theorem 2.1.  $\square$

**Remark 2.5.** If there is an index  $i$  such that  $d_i < k$ , then the mapping  $\Psi$  is not a submersion. However we can omit this problem as follows: Consider the mapping  $\Psi : \Omega_X(d_1, \dots, d_m) \times X \ni (F, x) \rightarrow j^k(F)(x) \in J^k(X, \mathbb{C}^m)$ . If we show that:

- 1)  $Y := \Psi(\Omega_X(d_1, \dots, d_m) \times X)$  is a smooth subvariety of  $J^2(X, \mathbb{C}^m)$ ,
- 2)  $S_I \cap \Psi(\Omega_X(d_1, \dots, d_m) \times X)$ , for every Thom-Boardman variety  $S_I$ ,
- 3) the mapping  $\Psi : \Omega_X(d_1, \dots, d_m) \times X \rightarrow Y$  is a submersion,

then we can proceed as above. We show that it is indeed the case at least when  $X = a \text{ plane}$  or  $X = a \text{ sphere}$ . However we believe that it is a general principle. We go back to this problem in the future.

### 3. PLANE MAPPINGS

Here we will study the set  $\Omega_2(d_1, d_2)$ . Let us denote coordinates in  $J^1(\mathbb{C}^2, \mathbb{C}^2)$  by

$$(x, y, f, g, f_x, f_y, g_x, g_y).$$

For a mapping  $F = (f, g) \in \Omega_2(d_1, d_2)$ , we have

$$j^1(F) = (x, y, f(x, y), g(x, y), \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y), \frac{\partial g}{\partial x}(x, y), \frac{\partial g}{\partial y}(x, y)),$$

which justifies our notation. The set  $S_1$  is given by the equation  $\phi(x, y, f, g, f_x, f_y, g_x, g_y) = f_x g_y - f_y g_x = 0$ . Since  $S_1$  describes elements of rank one it is easy to see that it is a smooth (non-closed) subvariety of  $J^1(\mathbb{C}^2, \mathbb{C}^2)$ .

Now we would like to describe the set  $S_{1,1}$  effectively. We restrict our attention only to sufficiently general jets. In the space  $J^2(\mathbb{C}^2, \mathbb{C}^2)$  we introduce coordinates

$$(x, y, f, g, f_x, f_y, g_x, g_y, f_{xx}, f_{yy}, f_{xy}, g_{xx}, g_{yy}, g_{xy}).$$

A generic mapping  $F$  satisfies  $\text{rank } d_a F \geq 1$  for every  $a$  (because  $\text{codim } S_2 = 4$ ). We can assume that  $F = (f, g)$  and  $\nabla_a f \neq 0$ . The critical set of  $F$  is exactly the set  $S_1(F)$  and it has a reduced equation  $\frac{\partial f}{\partial x}(x, y) \frac{\partial g}{\partial y}(x, y) - \frac{\partial f}{\partial y}(x, y) \frac{\partial g}{\partial x}(x, y) = 0$ , which by simplicity we write as  $f_x g_y - f_y g_x = 0$ . In particular the tangent line to  $S_1(f)$  is given as

$$(f_{xx} g_y + f_x g_{xy} - f_{xy} g_x - f_y g_{xx})v + (f_{xy} g_y + f_x g_{yy} - f_{yy} g_x - f_y g_{xy})w = 0.$$

Consequently the condition for  $[F_a] \in S_{1,1}$  is:

$$f_x g_y - f_y g_x = 0$$

and

$$(f_{xx} g_y + f_x g_{xy} - f_{xy} g_x - f_y g_{xx})f_y - (f_{xy} g_y + f_x g_{yy} - f_{yy} g_x - f_y g_{xy})f_x = 0.$$

Let us note that the last equation contains terms  $g_{xx} f_y^2$  and  $g_{yy} f_x^2$  hence for  $\nabla f \neq 0$  these two equations form a complete intersection. In general, if we omit the assumption  $\nabla f \neq 0$  the set  $S_{1,1}$  is given in  $J^2(\mathbb{C}^2, \mathbb{C}^2)$  by three equations:

$$L_1 := f_x g_y - f_y g_x = 0,$$

$$L_2 := (f_{xx} g_y + f_x g_{xy} - f_{xy} g_x - f_y g_{xx})f_y - (f_{xy} g_y + f_x g_{yy} - f_{yy} g_x - f_y g_{xy})f_x = 0,$$

and

$$L_3 := (f_{xx} g_y + f_x g_{xy} - f_{xy} g_x - f_y g_{xx})g_y - (f_{xy} g_y + f_x g_{yy} - f_{yy} g_x - f_y g_{xy})g_x = 0.$$

As above by symmetry the set  $S_{1,1}$  is smooth and locally is given as a complete intersection of either  $L_1, L_2$  or  $L_1, L_3$ .

We will denote by  $J, J_{1,1}, J_{1,2}$  curves given by  $L_1 \circ j^2(F) = 0$ ,  $L_2 \circ j^2(F) = 0$  and  $L_3 \circ j^2(F) = 0$ . We will also identify these curves with their equations.

**Remark 3.1.** These formulas give a description of  $S_{1,1}$  also in the case of a general affine surface  $X$ , however, it might be only locally in the Zariski topology of  $J^2(X, \mathbb{C}^2)$ .

**Definition 3.2.** Let  $F : (\mathbb{C}^2, a) \rightarrow (\mathbb{C}^2, F(a))$  be a holomorphic mapping. We say, that  $F$  has a fold at  $a$  if  $F$  is biholomorphically equivalent to the mapping  $(\mathbb{C}^2, 0) \ni (x, y) \mapsto (x, y^2) \in (\mathbb{C}^2, 0)$ . Moreover, we say that  $F$  has a simple cusp at  $a$  if  $F$  is biholomorphically equivalent to the mapping  $(\mathbb{C}^2, 0) \ni (x, y) \mapsto (x, y^3 + xy) \in (\mathbb{C}^2, 0)$ .

A direct consequence of Theorem 2.4 is:

**Theorem 3.3.** (cf. [17]) *Let  $X \subset \mathbb{C}^n$  be a smooth algebraic surface. Assume that  $d_1, d_2 > 1$ . Then there is a Zariski open subset  $U \subset \Omega_X(d_1, d_2)$  such that for every  $F \in U$  the mapping  $F$  has only folds and simple cusps as singularities.*

Now we compute the number of cusps of a generic polynomial mapping  $F \in \Omega_2(d_1, d_2)$ . To do this we need a series of lemmas:

**Lemma 3.4.** *Let  $L_\infty$  denote the line at infinity of  $\mathbb{C}^2$ . There is a non-empty open subset  $V \subset \Omega_2(d_1, d_2)$  such that for all  $(f, g) \in V$ :*

- (1)  $\overline{\left\{ \frac{\partial f}{\partial x} = 0 \right\}} \cap \overline{\left\{ \frac{\partial f}{\partial y} = 0 \right\}},$
- (2)  $\overline{\left\{ \frac{\partial f}{\partial x} = 0 \right\}} \cap \overline{\left\{ \frac{\partial f}{\partial y} = 0 \right\}} \cap L_\infty = \emptyset.$

*Proof.* The case  $d_1 = 1$  is trivial so assume  $d_1 > 1$ . Let us note that the set  $S \subset J^1(\mathbb{C}^2, \mathbb{C}^2)$  given by  $\{f_x = f_y = 0\}$  is smooth. Hence (1) follows from Theorem 2.1. To prove (2) it is enough to assume that  $f \in H_d$ , where  $H_d$  denotes the set of homogenous polynomials of two variables of degree  $d$ . Let  $\Psi : H_d \times (\mathbb{C} \times \mathbb{C}) \setminus \{0, 0\} \ni (f, x, y) \mapsto (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)) \in \mathbb{C}^2$ . It is easy to see that  $\Psi$  is a submersion. Indeed, if  $f = \sum a_i x^{d-i} y^i$  then  $f_x := \frac{\partial f}{\partial x}(x, y) = da_0 x^{d-1} + \dots + a_{d-1} y^{d-1}$ ,  $f_y := \frac{\partial f}{\partial y}(x, y) = a_1 x^{d-1} + \dots + da_d y^{d-1}$ . Since  $(x, y) \neq (0, 0)$  we can assume by symmetry that  $y \neq 0$ . Now  $\frac{\partial f_x}{\partial a_{d-1}} = y^{d-1}$ ,  $\frac{\partial f_x}{\partial a_d} = 0$ ,  $\frac{\partial f_y}{\partial a_d} = dy^{d-1}$ . Thus  $\frac{\partial(f_x, f_y)}{\partial(a_{d-1}, a_d)} = dy^{2(d-1)} \neq 0$ .

Hence for a generic polynomial  $f \in H_d$  the mapping  $\Psi_f : (\mathbb{C} \times \mathbb{C}) \setminus \{0, 0\} \ni (x, y) \mapsto (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)) \in \mathbb{C}^2$  is transversal to the point  $(0, 0)$ . In particular  $\Psi_f^{-1}(0, 0)$  is either zero-dimensional or the empty set. Since  $f$  is a homogenous polynomial the first possibility is excluded. This means that  $\overline{\left\{ \frac{\partial f}{\partial x} = 0 \right\}} \cap \overline{\left\{ \frac{\partial f}{\partial y} = 0 \right\}} \cap L_\infty = \emptyset$ .  $\square$

**Lemma 3.5.** *Let  $L_\infty$  denote the line at infinity of  $\mathbb{C}^2$ . There is a non-empty open subset  $V \subset \Omega_2(d_1, d_2)$  such that for all  $F = (f, g) \in V$ :*

- (1)  $\overline{J(F)} \cap \overline{J_{1,1}(F)} \cap L_\infty = \emptyset,$
- (2)  $\overline{J(F)} \cap L_\infty.$

*Proof.* Since the case  $d_1 = d_2 = 1$  is trivial we may assume that  $d_1 > 1$  or  $d_2 > 1$ . We consider the (generic) case when  $\deg f = d_1$  and  $\deg g = d_2$ . Hence  $\overline{J(F)} \cap L_\infty$



and  $\overline{J_{1,1}(F)} \cap L_\infty$  depend only on the homogeneous parts of  $f$  and  $g$  of degree  $d_1$  and  $d_2$  respectively. Let  $H_d$  denote the set of homogeneous polynomials of degree  $d$  in two variables. It is sufficient to show that there is an open subset  $V \subset H_{d_1, d_2} := H_{d_1} \times H_{d_2}$  such that  $\overline{J(F)} \cap \overline{J_{1,1}(F)} \cap L_\infty = \emptyset$  for all  $F = (f, g) \in V$ .

Consider the set  $X = \{(p, F) \in \mathbb{P}^1 \times H_{d_1, d_2} : J(F)(p) = J_{1,1}(F)(p) = 0\}$ . Note that  $X$  is a closed subset of  $\mathbb{P}^1 \times H_{d_1, d_2}$ , and if  $\overline{J(F)} \cap \overline{J_{1,1}(F)} \cap L_\infty \neq \emptyset$  then  $F$  belongs to the image of the projection of  $X$  on  $H_{d_1, d_2}$ . So to prove (1) it is sufficient to show that  $X$  has dimension strictly smaller than the dimension of  $H_{d_1, d_2}$ .

Let  $q = (1 : 0) \in \mathbb{P}^1$ ,  $Y := \{q\} \times H_{d_1, d_2}$  and  $X_0 = X \cap Y$ . Note that all fibers of the projection  $X \rightarrow \mathbb{P}^1$  are isomorphic to  $X_0$ . Thus  $\dim(X) = \dim(X_0) + \dim(\mathbb{P}^1)$  and to prove (1) it is sufficient to show that  $X_0$  has codimension at least 2 in  $Y$ .

Let  $p = (q, F) \in Y$  and let  $a_i$  and  $b_i$  be the parameters in  $H_{d_1, d_2}$  giving respectively the coefficients of  $f$  at  $x^{d_1-i}y^i$  and of  $g$  at  $x^{d_2-i}y^i$ . For  $0 \leq i + j \leq d_1$ , we have  $\frac{\partial^{i+j}f}{\partial x^i \partial y^j}(q) = \frac{(d_1-j)!j!}{(d_1-i-j)!}a_j(F)$  and similarly for  $g$  and  $b_j$ .

To conclude the proof of (1) we will show that the codimension of  $\{a_0b_0 = 0\} \cap X_0$  in  $Y$  is at least 2 and  $\nabla J$  and  $\nabla J_{1,1}$  are linearly independent outside  $\{a_0b_0 = 0\} \cap X_0$  and thus the variety  $X_0$  has codimension 2 in  $Y$ .

Let us calculate  $J(p)$ . We have  $J(p) = (f_x g_y - f_y g_x)(q, F) = (d_1 a_0 b_1 - d_2 a_1 b_0)(F)$ . Thus  $\{a_0 = 0\} \cap X_0 \subset \{a_0 = a_1 b_0 = 0\} \cap Y$  has codimension at least 2 and we may assume in further calculations that  $a_0(F) \neq 0$  and similarly  $b_0(F) \neq 0$ .

Let us assume that  $d_2 > 1$ , we have  $\frac{\partial J}{\partial b_1}(p) = \frac{\partial d_1 a_0 b_1 - d_2 a_1 b_0}{\partial b_1}(F) = d_1 a_0(F)$  and  $\frac{\partial J(p)}{\partial b_2} = 0$ . Now let us calculate  $\frac{\partial J_{1,1}}{\partial b_2}(p)$ . The coefficient  $b_2$  can only be obtained from  $\frac{\partial^2 g}{\partial y^2}$ , which is present in  $J_{1,1}$  in the summand  $-2 \frac{\partial^2 g}{\partial y^2} (d_1 \frac{\partial f}{\partial x})^2$ . Thus  $\frac{\partial J_{1,1}}{\partial b_2}(p) = \frac{\partial(-2d_1^2 b_2 a_0^2)}{\partial b_2}(F) = -2(d_1 a_0(F))^2$ . So  $\det \frac{\partial(J, J_{1,1})}{\partial(b_1, b_2)}(p) = -2(d_1 a_0(F))^3 \neq 0$ .

Similarly, if  $d_2 = 1$  and  $d_1 > 1$  then  $\det \frac{\partial(J, J_{1,1})}{\partial(a_1, a_2)}(p) = -2(d_1 a_0(F))(d_2 b_0(F))^2 \neq 0$ .

To prove (2) note that  $\overline{\{\frac{\partial J}{\partial x}(F) = 0\}} \cap \overline{\{\frac{\partial J}{\partial y}(F) = 0\}} \subset \overline{J_{1,1}(F)}$ , hence (1) implies (2).  $\square$

**Lemma 3.6.** *There is a non-empty open subset  $V_1 \subset \Omega_2(d_1, d_2)$  such that for all  $(f, g) \in V_1$  and every  $a \in \mathbb{C}^2$ : if  $\frac{\partial f}{\partial x}(a) = 0$  and  $\frac{\partial f}{\partial y}(a) = 0$ , then  $\frac{\partial g}{\partial x}(a) \neq 0$  and  $\frac{\partial g}{\partial y}(a) \neq 0$ .*

*Proof.* Let us consider two subsets in  $J^1(\mathbb{C}^2, \mathbb{C}^2)$ :  $R_1 := \{(x, y, f, g, f_x, f_y, g_x, g_y) : f_x = 0, f_y = 0, g_x = 0\}$  and  $R_2 := \{(x, y, f, g, f_x, f_y, g_x, g_y) : f_x = 0, f_y = 0, g_y = 0\}$ . By Theorem 2.1 there is a non-empty open subset  $V_1 \subset \Omega_2(d_1, d_2)$  such that for every  $F \in V_1$  the mapping  $j^1(F)$  is transversal to  $R_1$  and  $R_2$ . Since these subsets have codimension three, we see that the image of  $j^1(F)$  is disjoint with  $R_1$  and  $R_2$ .  $\square$

**Lemma 3.7.** *There is a non-empty open subset  $V_2 \subset \Omega_2(d_1, d_2)$  such that for all  $(f, g) \in V_2$  we have  $\{\frac{\partial f}{\partial x} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\} \cap J_{1,2}(f, g) = \emptyset$ .*

*Proof.* Let us consider the (non-closed) subvariety  $S \subset J^2(2)$  given by equations:  $f_x = 0$ ,  $f_y = 0$ ,  $(f_{xx}g_y + f_x g_{xy} - f_{xy}g_x - f_y g_{xx})g_y - (f_{xy}g_y + f_x g_{yy} - f_{yy}g_x - f_y g_{xy})g_x = 0$ ,  $g_x \neq 0$ ,  $g_y \neq 0$ . It is easy to check that  $S$  is a smooth complete intersection and it has codimension three. The set of generic mappings  $F$  which are transversal to  $S$  contains a Zariski open dense subset  $V_2 \subset \Omega_2(d_1, d_2)$ . By construction for all  $(f, g) \in V_2$  we have  $\{\frac{\partial f}{\partial x} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\} \cap J_{1,2}(f, g) = \emptyset$ .  $\square$

**Lemma 3.8.** *There is a non-empty open subset  $V_3 \subset \Omega_2(d_1, d_2)$  such that for all  $(f, g) \in V_3$  the curve  $J(f, g)$  is transversal to the curve  $J_{1,1}(f, g)$ .*

*Proof.* There is a Zariski open subset  $V_3$  which contains only generic mappings which satisfy hypotheses of all lemmas above. We can also assume that the curves  $\{\frac{\partial f}{\partial x} = 0\}$  and  $\{\frac{\partial f}{\partial y} = 0\}$  intersect transversally. We have to show that the curves  $J(f, g)$  and  $J_{1,1}(f, g)$  intersect transversally at every point  $a \in J(f, g) \cap J_{1,1}(f, g)$ . If  $\nabla f \neq 0$  then it follows from transversality of the mapping  $F$  to the set  $S_{1,1}$ . Hence we can assume  $\{\frac{\partial f}{\partial x}(a) = 0\}$  and  $\{\frac{\partial f}{\partial y}(a) = 0\}$ . By Lemma 3.6 we have  $\frac{\partial g}{\partial x}(a) \neq 0$  and  $\frac{\partial g}{\partial y}(a) \neq 0$ . Let us denote:  $\frac{\partial f}{\partial x}(x, y) = f_x$ ,  $\frac{\partial f}{\partial y}(x, y) = f_y$ , etc. It is enough to prove that in the ring  $\mathcal{O}_a^2$  we have the equality  $I = (f_x g_y - f_y g_x, (f_{xx} g_y + f_{xy} g_x - f_{xy} g_x - f_y g_{xx}) f_y - (f_{xy} g_y + f_{xx} g_{yy} - f_{yy} g_x - f_y g_{xy}) f_x) = \mathfrak{m}_a$ , where  $\mathfrak{m}_a$  denotes the maximal ideal of  $\mathcal{O}_a^2$ . Put  $L = f_x g_y - f_y g_x$ . Hence  $I = (L, L_x f_y - L_y f_x)$ . Since  $g_x(a) \neq 0, g_y(a) \neq 0$ , we have

$$\begin{aligned} I &= (L, g_x[L_x f_y - L_y f_x], g_y[L_x f_y - L_y f_x]) = (L, L_x g_x f_y - L_y g_x f_x, L_x g_y f_y - L_y g_y f_x) = \\ &= (L, L_x g_y f_x - L_y g_x f_x, L_x g_y f_y - L_y g_x f_y) = (L, f_x[L_x g_y - L_y g_x], f_y[L_x g_y - L_y g_x]). \end{aligned}$$

By Lemma 3.7 we have  $[L_x g_y - L_y g_x](a) \neq 0$ , hence  $I = (f_x, f_y) = \mathfrak{m}_a$ .  $\square$

Now we are in a position to prove:

**Theorem 3.9.** *There is a Zariski open, dense subset  $U \subset \Omega_2(d_1, d_2)$  such that for every mapping  $F \in U$  the mapping  $F$  has only two-folds and cusps as singularities and the number of cusps is equal to*

$$d_1^2 + d_2^2 + 3d_1 d_2 - 6d_1 - 6d_2 + 7.$$

Moreover, if  $d_1 > 1$  or  $d_2 > 1$  then the set  $C(F)$  of critical points of  $F$  is a smooth connected curve, which is topologically equivalent to a sphere with  $g = \frac{(d_1 + d_2 - 3)(d_1 + d_2 - 4)}{2}$  handles and  $d_1 + d_2 - 2$  points removed.

*Proof.* If  $d_1 = d_2 = 1$  then the theorem is obvious. Hence we can assume that  $d_1 > 1$ . Assume first that also  $d_2 > 1$ . Note that every point  $a$  of the intersection of curves  $J(f, g)$  and  $J_{1,1}(f, g)$  with  $\nabla_a f \neq 0$  is a cusp. Moreover for a generic mapping  $F$  points with  $\nabla_a f = 0$  are not cusps (Lemma 3.8). By Bezout Theorem we have that in  $J(f, g) \cap J_{1,1}(f, g)$  there are exactly  $(d_1 - 1)^2$  points with  $\nabla f = 0$  and that the number of cusps of a generic mapping is equal to

$$(d_1 + d_2 - 2)(2d_1 + d_2 - 4) - (d_1 - 1)^2 = d_1^2 + d_2^2 + 3d_1 d_2 - 6d_1 - 6d_2 + 7.$$

If  $d_2 = 1$  then we can replace the space  $J^2(\mathbb{C}^2, \mathbb{C}^2)$  by its subspace  $Y$  given by equations  $g_{xx} = g_{xy} = g_{yy} = 0$ . Note that varieties  $S_1, S_{1,1}$  are transversal to  $Y$ . Moreover the mapping  $\Psi : \Omega_2(d, 1) \times \mathbb{C}^2 \rightarrow Y$  is a submersion and we can proceed as above. We leave the details to the reader.

Finally by Lemma 3.5 we have that  $C(F) = S_1(F)$  is a smooth affine curve which is transversal to the line at infinity. This means that  $\overline{C(F)}$  is also smooth at infinity, hence it is a smooth projective curve of degree  $d = d_1 + d_2 - 2$ . Thus by the Riemann-Roch Theorem the curve  $\overline{C(F)}$  has genus  $g = \frac{(d-1)(d-2)}{2}$ . This means in particular that  $\overline{C(F)}$  is homeomorphic to a sphere with  $g = \frac{(d-1)(d-2)}{2}$  handles. Moreover, by the Bezout Theorem it has precisely  $d$  points at infinity.  $\square$



## 4. THE DISCRIMINANT

Here we analyze the discriminant of a generic mapping from  $\Omega(d_1, d_2)$ . Let us recall that the discriminant of the mapping  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is the curve  $\Delta(F) := F(C(F))$ , where  $C(F)$  is the critical curve of  $F$ . We have:

**Lemma 4.1.** *There is a non-empty open subset  $U \subset \Omega_2(d_1, d_2)$  such that for every mapping  $F \in U$ :*

- (1)  $F|_{C(F)}$  is injective outside a finite set,
- (2) if  $p \in \Delta(F)$  then  $|F^{-1}(p) \cap C(F)| \leq 2$ ,
- (3) if  $|F^{-1}(p) \cap C(F)| = 2$  then the curve  $\Delta(F)$  has a normal crossing at  $p$ .

*Proof.* We may assume that  $d_1 \geq d_2$ . Let  $\Omega_2^*(d_1, d_2)$  be the set of  $(f, g) \in \Omega_2(d_1, d_2)$  such that  $g - g(0, 0)$  is not 0 and does not divide  $f - f(0, 0)$ . Note that  $\Omega_2^*(d_1, d_2)$  is a non-empty open subset of  $\Omega_2(d_1, d_2)$  and if  $F \in \Omega_2^*(d_1, d_2)$  and  $\alpha \in \Omega_2(1, 1)$  is an affine automorphism of  $\mathbb{C}^2$  then  $F \circ \alpha \in \Omega_2^*(d_1, d_2)$ .

To prove (1) consider the set  $X = \{(p, q, F) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \Omega_2^*(d_1, d_2) : p \neq q, F(p) = F(q), J(F)(p) = J(F)(q) = 0\}$ . We will show that  $X$  has dimension not greater than  $\dim \Omega_2^*(d_1, d_2)$ . So the projection of  $X$  on  $\Omega_2^*(d_1, d_2)$  has finite fibers on some open subset  $U \subset \Omega_2^*(d_1, d_2)$ . Moreover if the fiber over  $F$  is finite then  $F|_{C(F)}$  is injective outside a finite set given by the fiber.

Let  $p = (0, 0)$ ,  $q = (0, 1)$ ,  $Y := \{p\} \times \{q\} \times \Omega_2^*(d_1, d_2)$  and  $X_0 = X \cap Y$ . Note that all fibers of the projection  $X \rightarrow \mathbb{C}^2 \times \mathbb{C}^2$  are isomorphic to  $X_0$ . Thus  $\dim(X) = \dim(X_0) + 4$  and to prove (1) it is sufficient to show that  $X_0$  has codimension at least 4 in  $Y$ .

Let  $(p, q, F) \in Y$  and let  $a_{ij}$  and  $b_{ij}$  be the parameters in  $\Omega_2(d_1, d_2)$  giving respectively the coefficients of  $f$  and  $g$  at  $x^i y^j$ . For  $0 \leq i + j \leq d_1$ , we have  $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(p) = i!j!a_{ij}(F)$  and  $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(q) = i! \sum_{k=j}^{d_1-i} \frac{k!}{(k-j)!} a_{ik}(F)$  and similarly for  $g$  and  $b_{ij}$ .

The condition  $F(p) = F(q)$  yields the equations  $w_1 = \sum_{j=1}^{d_1} a_{0j}(F) = 0$  and  $w_2 = \sum_{j=1}^{d_2} b_{0j}(F) = 0$ , the conditions  $J(F)(p) = J(F)(q) = 0$  give  $w_3 = (a_{10}b_{01} - a_{01}b_{10})(F) = 0$  and  $w_4 = (\sum_{j=0}^{d_1-1} a_{1j} \sum_{j=1}^{d_2} j b_{0j} - \sum_{j=1}^{d_1} j a_{0j} \sum_{j=0}^{d_2-1} b_{1j})(F) = 0$ . If  $d_2 \geq 2$  then note that the matrix  $\frac{\partial(w_1, w_2, w_3, w_4)}{\partial(a_{01}, b_{01}, a_{10}, a_{11})}$  is triangular and its determinant is equal to  $b_{01}(F) \sum_{j=1}^{d_2} b_{0j}(F)$ . Calculating similar derivations with  $a_{10}$  replaced by  $b_{10}$  or  $a_{11}$  replaced by  $b_{11}$  we obtain that  $\nabla w_1, \dots, \nabla w_4$  are independent outside  $S = \{a_{01}(F) = b_{01}(F) = 0\} \cup \{\sum_{j=1}^{d_2} a_{0j}(F) = \sum_{j=1}^{d_2} b_{0j}(F) = 0\}$ . Thus  $X_0 \setminus S$  has codimension 4 in  $Y$  and it is easy to see that  $X_0 \cap S$  has also codimension at least 4.

If  $d_2 = 1$  then we have  $w_2 = b_{01}(F) = 0$ . Since  $F = (f, g) \in \Omega_2^*(d_1, 1)$  we have  $b_{10}(F) \neq 0$  and  $x$  does not divide  $f - f(0, 0)$ , i.e.  $a_{0j}(F) \neq 0$  for some  $j \geq 1$ . Moreover we may take  $w_3 = a_{01}(F) = 0$  and  $w_4 = \sum_{j=1}^{d_1} j a_{0j}(F) = 0$ . If  $d_1 = 2$  then we obtain a contradiction thus showing that  $X$  is in fact empty. If  $d_1 \geq 3$  then calculating  $\det \frac{\partial(w_1, w_2, w_3, w_4)}{\partial(b_{01}, a_{01}, a_{02}, a_{03})} = 1$  we obtain that  $X_0$  has codimension 4.

To prove (2) consider the set  $X = \{(p, q, r, F) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \Omega_2^*(d_1, d_2) : p \neq q \neq r \neq p, F(p) = F(q) = F(r), J(F)(p) = J(F)(q) = J(F)(r) = 0\}$ . Similarly as in (1) we compute that  $X$  has codimension at least 7. It follows that the projection of  $X$  on  $\Omega_2^*(d_1, d_2)$  has empty fibers on some open subset  $U \subset \Omega_2^*(d_1, d_2)$ . Note that unlike in (1) there are two types of fibers of the projection onto  $\mathbb{C}^6$ :  $X_0 := \{(0, 0), (1, 0), (0, 1)\} \times \Omega_2^*(d_1, d_2) \cap X$

and  $X_t := \{((0, 0), (0, 1), (0, t))\} \times \Omega_2^*(d_1, d_2) \cap X$ . In both cases the computation is purely technical and similar to the computation in (1), so we leave the details to the reader.

To prove (3) note that if  $q \in C(F)$  then  $d_q F(T_q C(F))$  is spanned by the vector  $(J_{1,1}(F)(q), J_{1,2}(F)(q))$ . Thus if  $F^{-1}(p) \cap C(F) = \{q_1, q_2\}$  then  $\Delta(F)$  has a normal crossing at  $p$  if and only if  $(J_{1,1}(F)(q_1), J_{1,2}(F)(q_1))$  and  $(J_{1,1}(F)(q_2), J_{1,2}(F)(q_2))$  are independent, i.e.  $J_{1,1}(F)(q_1)J_{1,2}(F)(q_2) - J_{1,2}(F)(q_1)J_{1,1}(F)(q_2) \neq 0$ . Similarly as in (1) let us consider the set  $X = \{(p, q, F) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \Omega_2^*(d_1, d_2) : p \neq q, F(p) = F(q), J(F)(p) = J(F)(q) = J_{1,1}(F)(p)J_{1,2}(F)(q) - J_{1,2}(F)(p)J_{1,1}(F)(q) = 0\}$ . One can compute that  $X$  has codimension at least 5, thus the projection of  $X$  on  $\Omega_2^*(d_1, d_2)$  has empty fibers on some open subset  $U \subset \Omega_2^*(d_1, d_2)$ .  $\square$

Hence for a generic  $F$  the only singularities of  $\Delta(F)$  are cusps and nodes. We showed in Theorem 3.9 that there are exactly  $c(F) = d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$  cusps. Now we will compute the number  $d(F)$  of nodes of  $\Delta(F)$ . We will use the following theorem of Serre (see [13], p. 85):

**Theorem 4.2.** *If  $\Gamma$  is an irreducible curve of degree  $d$  and genus  $g$  in the complex projective plane then*

$$\frac{1}{2}(d-1)(d-2) = g + \sum_{z \in \text{Sing}(\Gamma)} \delta_z,$$

where  $\delta_z$  denotes the delta invariant of a point  $z$ .

First we compute the degree of the discriminant:

**Lemma 4.3.** *Let  $F = (f, g) \in \Omega(d_1, d_2)$  be a generic mapping. If  $d_1 \geq d_2$  then  $\deg \Delta(F) = d_1(d_1 + d_2 - 2)$ .*

*Proof.* Let  $L \subset \mathbb{C}^2$  be a generic line  $\{ax + by + c = 0\}$ . Then  $L$  intersects  $\Delta(F)$  in smooth points and  $\deg \Delta(F) = \#L \cap \Delta(F)$ . If  $j : C(F) \rightarrow \Delta(F)$  is a mapping induced by  $F$  then  $\#L \cap \Delta(F) = \#j^{-1}(L \cap \Delta(F))$ . The curve  $j^{-1}(L) = \{af + bg + c = 0\}$  has no common points at infinity with  $C(F)$ . Hence by Bezout Theorem we have  $\#j^{-1}(L \cap \Delta(F)) = (\deg j^{-1}(L))(\deg C(F)) = d_1(d_1 + d_2 - 2)$ . Consequently  $\deg \Delta(F) = d_1(d_1 + d_2 - 2)$ .  $\square$

We have the following method of computing the delta invariant (see [13], p. 92-93):

**Theorem 4.4.** *Let  $V_0 \subset \mathbb{C}^2$  be an irreducible germ of an analytic curve with the Puiseux parametrization of the form*

$$z_1 = t^{a_0}, \quad z_2 = \sum_{i>0} \lambda_i t^{a_i}, \quad \text{where } \lambda_i \neq 0, \quad a_1 < a_2 < a_3 < \dots$$

Let  $D_j = \gcd(a_0, a_1, \dots, a_{j-1})$ . Then

$$\delta_0 = \frac{1}{2} \sum_{j \geq 1} (a_j - 1)(D_j - D_{j+1}).$$

If  $V = \bigcup_{i=1}^r V_i$  has  $r$  branches then

$$\delta(V) = \sum_{i=1}^r \delta(V_i) + \sum_{i < j} V_i \cdot V_j,$$

where  $V \cdot W$  denotes the intersection product.

The main result of this section will be based on the following:

**Theorem 4.5.** *Let  $F \in \Omega(d_1, d_2)$  be a generic mapping. Let  $d_1 \geq d_2$  and  $d = \gcd(d_1, d_2)$ . Denote by  $\overline{\Delta}$  the projective closure of the discriminant  $\Delta$ . Then*

$$\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = \frac{1}{2} d_1 (d_1 - d_2) (d_1 + d_2 - 2)^2 + \frac{1}{2} (-2d_1 + d_2 + d) (d_1 + d_2 - 2).$$

*Proof.* Let  $\tilde{f}(x, y, z) = z^{d_1} f(\frac{x}{z}, \frac{y}{z})$  and  $\tilde{g}(x, y, z) = z^{d_2} g(\frac{x}{z}, \frac{y}{z})$  be the homogenizations of  $f$  and  $g$  and let  $\overline{f}(x, z) = \tilde{f}(x, 1, z)$  and  $\overline{g}(x, z) = \tilde{g}(x, 1, z)$ . For a generic mapping the curves  $C(F)$  and  $\{f = 0\}$  have no common points at infinity (see Lemma 4.6). Moreover we may assume that  $(1 : 0 : 0) \notin \overline{C(F)}$ . Thus  $F$  extends to a neighborhood of  $\overline{C(F)} \cap L_\infty$  on which it is given by the formula

$$\overline{F}(x, z) = \left( z^{d_1 - d_2} \frac{\overline{g}(x, z)}{\overline{f}(x, z)}, \frac{z^{d_1}}{\overline{f}(x, z)} \right).$$

Let  $L_\infty^* = L_\infty \setminus \{(1 : 0 : 0)\}$  and  $\{P_1, \dots, P_{d_1 + d_2 - 2}\} = \overline{C(F)} \cap L_\infty^*$ , fix a point  $P = P_i$ . The curve  $\overline{C(F)}$  is transversal to the line at infinity so it has a local parametrization at  $P$  of the form  $\gamma(t) := (\sum_i a_i t^i, t)$ . We have the following:

**Lemma 4.6.** *If  $F$  is a generic mapping then  $\overline{f}(P) \neq 0$ ,  $\overline{g}(P) \neq 0$  and*

$$\overline{f}(\gamma(t)) = \overline{f}(P)(1 + ct + \dots), \quad \overline{g}(\gamma(t)) = \overline{g}(P)(1 + dt + \dots),$$

where  $cd \neq 0$  and  $d_2 c \neq d_1 d$ .

*Proof.* Let  $\tilde{J} = \tilde{J}(F)$  be the homogenization of  $J(F)$ . Obviously  $\tilde{J} = \frac{\partial \tilde{f}}{\partial x} \frac{\partial \tilde{g}}{\partial y} - \frac{\partial \tilde{f}}{\partial y} \frac{\partial \tilde{g}}{\partial x}$ . Let  $\overline{J}(x, z) = \tilde{J}(x, 1, z)$ . Since  $\overline{J}(\gamma(t)) = 0$  and  $\frac{\partial \gamma(t)}{\partial t}|_{t=0} = (a_1, 1)$  we have

$$\begin{aligned} \overline{f}(P) c \frac{\partial \overline{J}(P)}{\partial x} &= \left( \frac{\partial \overline{f}(\gamma(t))}{\partial t} \frac{\partial \overline{J}(\gamma(t))}{\partial x} \right)_{|t=0} = \\ &= \left( \frac{\partial \overline{f}(\gamma(t))}{\partial x} a_1 \frac{\partial \overline{J}(\gamma(t))}{\partial x} + \frac{\partial \overline{f}(\gamma(t))}{\partial z} \frac{\partial \overline{J}(\gamma(t))}{\partial x} \right)_{|t=0} = \\ &= \left( -\frac{\partial \overline{f}(\gamma(t))}{\partial x} \frac{\partial \overline{J}(\gamma(t))}{\partial z} + \frac{\partial \overline{f}(\gamma(t))}{\partial z} \frac{\partial \overline{J}(\gamma(t))}{\partial x} \right)_{|t=0} = \frac{\partial \overline{f}(P)}{\partial z} \frac{\partial \overline{J}(P)}{\partial x} - \frac{\partial \overline{f}(P)}{\partial x} \frac{\partial \overline{J}(P)}{\partial z} \end{aligned}$$

Consider the set

$$X = \left\{ (p, F) \in L_\infty^* \times \Omega_2(d_1, d_2) : \tilde{J}(F)(p) = \left( \frac{\partial \tilde{f}}{\partial z} \frac{\partial \tilde{J}(F)}{\partial x} - \frac{\partial \tilde{f}}{\partial x} \frac{\partial \tilde{J}(F)}{\partial z} \right)(p) = 0 \right\}.$$

Note that if  $\overline{f}(P) = 0$  or  $c = 0$  then the fiber over  $P$  of the projection from  $X$  to  $\Omega_2(d_1, d_2)$  is non-empty. Hence it suffices to prove that  $X$  has codimension at least 2.

Let  $p = (0 : 1 : 0)$ , and  $q = (a : b : 0) \in L_\infty^*$ . Let  $T(x, y, z) = (bx - ay, y, z)$  so that  $T(q) = p$ . Note that  $\tilde{J}(F \circ T) = (\tilde{J}(F) \circ T)J(T) = b\tilde{J}(F) \circ T$ . Furthermore

$$\frac{\partial \tilde{f} \circ T}{\partial z} \frac{\partial \tilde{J}(F \circ T)}{\partial x} - \frac{\partial \tilde{f} \circ T}{\partial x} \frac{\partial \tilde{J}(F \circ T)}{\partial z} = b^2 \left( \frac{\partial \tilde{f}}{\partial z} \frac{\partial \tilde{J}(F)}{\partial x} - \frac{\partial \tilde{f}}{\partial x} \frac{\partial \tilde{J}(F)}{\partial z} \right) \circ T$$

Thus  $(p, F) \mapsto (T^{-1}(p), F \circ T)$  is an isomorphism of  $X_p := X \cap (\{p\} \times \Omega_2(d_1, d_2))$  and  $X \cap (\{q\} \times \Omega_2(d_1, d_2))$ . So it is enough to show that  $X_p$  has codimension 2 in  $Y_p := \{p\} \times \Omega_2(d_1, d_2)$ .

Let  $a_i$  be the parameters in  $\Omega_2(d_1, d_2)$  giving the coefficients of  $\tilde{f}$  (and of  $f$ ) at  $x^{d_1-i}y^i$  and let  $b_i$  and  $c_i$  describe respectively the coefficients of  $\tilde{g}$  at  $x^{d_2-i}y^i$  and  $x^{d_2-i-1}y^iz$ .

The first equation of  $X_p$  is  $d_2a_{d_1-1}b_{d_2} - d_1a_{d_1}b_{d_2-1} = 0$  and the only summand of the second containing  $c_{d_2-1}$  is  $(a_{d_1-1})^2(d_2-1)c_{d_2-1}$ . Clearly those equations are independent outside the set  $\{a_{d_1-1} = 0\}$ . Moreover  $\{a_{d_1-1} = d_2a_{d_1-1}b_{d_2} - d_1a_{d_1}b_{d_2-1} = 0\} = \{a_{d_1-1} = a_{d_1} = 0\} \cup \{a_{d_1-1} = b_{d_2-1} = 0\}$ , thus  $X_p$  has codimension 2 in  $Y_p$ .

Finally note that if  $d_2c = d_1d$  then

$$d_2\bar{g}(P) \left( \frac{\partial \bar{f}}{\partial z} \frac{\partial \bar{J}}{\partial x} - \frac{\partial \bar{f}}{\partial x} \frac{\partial \bar{J}}{\partial z} \right) (P) = d_1\bar{f}(P) \left( \frac{\partial \bar{g}}{\partial z} \frac{\partial \bar{J}}{\partial x} - \frac{\partial \bar{g}}{\partial x} \frac{\partial \bar{J}}{\partial z} \right) (P).$$

Hence we consider the set

$$X = \left\{ (p, F) \in L_\infty \times \Omega_2(d_1, d_2) : \tilde{J}(F)(p) = d_2\tilde{g}(p) \left( \frac{\partial \tilde{f}}{\partial z} \frac{\partial \tilde{J}}{\partial x} - \frac{\partial \tilde{f}}{\partial x} \frac{\partial \tilde{J}}{\partial z} \right) (p) - d_1\tilde{f}(p) \left( \frac{\partial \tilde{g}}{\partial z} \frac{\partial \tilde{J}}{\partial x} - \frac{\partial \tilde{g}}{\partial x} \frac{\partial \tilde{J}}{\partial z} \right) (p) = 0 \right\}.$$

Similarly as above one can show that it has codimension 2, which concludes the proof.  $\square$

Let  $C_p$  be the branch of  $\overline{C(F)}$  at  $P$ . We find the Puiseux expansion of the branch  $\overline{F}(C_P)$  of  $\overline{\Delta(F)}$  at  $\overline{F}(P)$ . We have

$$\begin{aligned} \overline{F}(\gamma(t)) &= \left( t^{d_1-d_2} \frac{\bar{g}(\gamma(t))}{\bar{f}(\gamma(t))}, \frac{t^{d_1}}{\bar{f}(\gamma(t))} \right) = \\ &= \left( t^{d_1-d_2}(1 + (d-c)t + \dots) \frac{\bar{g}(P)}{\bar{f}(P)}, \frac{t^{d_1}(1 - ct + \dots)}{\bar{f}(P)} \right). \end{aligned}$$

If  $d_1 = d_2$  then by Lemma 4.6 we have  $d - c \neq 0$  and  $\overline{F}(C_P)$  is smooth at  $\overline{F}(P)$ . So assume  $d_1 > d_2$ . Since the function  $h(t) = \left( \frac{\bar{f}(P)}{\bar{g}(P)} \frac{\bar{g}(\gamma_P(t))}{\bar{f}(\gamma_P(t))} \right)^{\frac{1}{d_1-d_2}} = 1 + \frac{d-c}{d_1-d_2}t + \dots$  is invertible in  $t = 0$  we can introduce a new variable  $T = th(t)$ . We have  $\overline{F}(\gamma(T)) = \left( T^{d_1-d_2} \frac{\bar{g}(P)}{\bar{f}(P)}, T^{d_1} h(t)^{-d_1} (1 - ct + \dots) \frac{1}{\bar{f}(P)} \right)$ . Moreover  $h(t)^{-d_1} (1 - ct + \dots) = (1 - d_1 \frac{d-c}{d_1-d_2} T + \dots)(1 - cT + \dots) = 1 + \frac{d_2c-d_1d}{d_1-d_2}T + \dots$ . By Lemma 4.6 we have  $d_2c - d_1d \neq 0$  and we can apply Theorem 4.4 to compute  $\delta(C_P)_{\overline{F}(P)}$ . Since  $a_0 = d_1 - d_2$ ,  $a_1 = d_1$  and  $a_2 = d_1 + 1$ , we have  $2\delta(C_P)_{\overline{F}(P)} = (d_1 - 1)(d_1 - d_2 - d) + (d_1 + 1 - 1)(d - 1) = (d_1 - 1)(d_1 - d_2 - 1) + (d - 1)$ , where  $d = \gcd(d_1, d_2)$ .

To proceed further we also need:

**Lemma 4.7.** *If  $F$  is a generic mapping then*

$$\bar{f}(P_i)^{d_2} \bar{g}(P_j)^{d_1} \neq \bar{f}(P_j)^{d_2} \bar{g}(P_i)^{d_1}$$

for  $i, j \in \{1, 2, \dots, d_1 + d_2 - 2\}$  and  $i \neq j$ .

*Proof.* Consider the set  $X = \{(p, q, F) \in L_\infty \times L_\infty \times \Omega_2(d_1, d_2) : p \neq q, \tilde{J}(F)(p) = \tilde{J}(F)(q) = \tilde{f}(p)^{d_2} \tilde{g}(q)^{d_1} - \tilde{f}(q)^{d_2} \tilde{g}(p)^{d_1} = 0\}$ . Similarly as in Lemma 4.6 we will prove that  $X$  has codimension 3, so there is a dense open subset  $S \subset \Omega(d_1, d_2)$  such that the projection from  $X$  has empty fibers over  $F \in S$ .

Indeed, take  $p = (1 : 0 : 0)$ ,  $q = (0 : 1 : 0)$  and  $Y := \{(p, q)\} \times \Omega_2(d_1, d_2)$ , it suffices to show that  $X_0 = X \cap Y$  has codimension 3 in  $Y$ . Let  $a_i$  and  $b_i$  be the parameters in  $\Omega_2(d_1, d_2)$  giving respectively the coefficients of  $\tilde{f}$  at  $x^{d_1-i}y^i$  and of  $\tilde{g}$  at  $x^{d_2-i}y^i$ .

The three equations describing  $X_0$  are  $w_1 = d_1 a_0 b_1 - d_2 a_1 b_0 = 0$ ,  $w_2 = d_2 a_{d_1-1} b_{d_2} - d_1 a_{d_1} b_{d_2-1} = 0$  and  $w_3 = a_0^{d_2} b_{d_2}^{d_1} - a_{d_1}^{d_2} b_0^{d_1} = 0$ . Note that  $X_0 \cap \{a_0 = 0\} = \{a_0 = b_0 = w_2 = 0\} \cup \{a_0 = a_1 = a_{d_1} = w_2 = 0\}$  has codimension 3. Similarly  $X_0 \cap \{b_0 = 0\}$  and  $X_0 \cap \{a_{d_1} = 0\}$  have codimension 3, however outside the set  $\{a_0 = b_0 = a_{d_1}\}$  the three equations are obviously independent. Thus  $X_0$  has codimension 3 in  $X$ .  $\square$

Now we are in a position to compute  $\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z$ . If  $d_1 = d_2$  then  $\overline{\Delta}$  has exactly  $d_1 + d_2 - 2$  smooth points at infinity and consequently  $\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = 0$ . So assume  $d_1 > d_2$ , then  $\overline{\Delta}$  has only one point at infinity  $Q = (1 : 0 : 0)$ . In  $Q$  the curve  $\overline{\Delta}$  has exactly  $r = d_1 + d_2 - 2$  branches  $V_i = \overline{F}(C_{P_i})$ . We computed above that  $2\delta(V_i)_Q = (d_1 - 1)(d_1 - d_2 - 1) + (d - 1)$ . Now we will compute  $V_i \cdot V_j$ . Let  $t_{a,b}(x, y) = (x + a, y + b)$ . By the dynamical definition of intersection there exists a neighborhood  $U$  of 0, such that for small generic  $a, b$  we have

$$V_i \cdot V_j = \#(U \cap V_i \cap t_{a,b}(V_j)).$$

This means that  $V_i \cdot V_j$  is equal to the number of solutions of the following system:

$$\frac{\overline{g}(P_i)}{\overline{f}(P_i)} T^{d_1-d_2} = \frac{\overline{g}(P_j)}{\overline{f}(P_j)} S^{d_1-d_2} + a,$$

$$\frac{1}{\overline{f}(P_i)} T^{d_1} (1 + \alpha_i T + \dots) = \frac{1}{\overline{f}(P_j)} S^{d_1} (1 + \alpha_j S + \dots) + b,$$

where  $a, b$  and  $S, T$  are sufficiently small. Take

$$Q : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0),$$

$$Q(T, S) = \left( \frac{\overline{g}(P_i)}{\overline{f}(P_i)} T^{d_1-d_2} - \frac{\overline{g}(P_j)}{\overline{f}(P_j)} S^{d_1-d_2}, \frac{1}{\overline{f}(P_i)} T^{d_1} (1 + \alpha_i T + \dots) - \frac{1}{\overline{f}(P_j)} S^{d_1} (1 + \alpha_j S + \dots) \right).$$

Thus we have  $V_i \cdot V_j = \text{mult}_0 Q$ . Note that by Lemma 4.7 the minimal homogenous polynomials of the two components of  $Q$  have no nontrivial common zeroes, hence  $V_i \cdot V_j = d_1(d_1 - d_2)$ . Consequently

$$\begin{aligned} \sum_i \delta(V_i) + \sum_{i>j} V_i \cdot V_j &= \frac{1}{2} [(d_1 - 1)(d_1 - d_2 - 1) + (d - 1)](d_1 + d_2 - 2) + \\ &\quad \frac{1}{2} d_1(d_1 - d_2)(d_1 + d_2 - 2)(d_1 + d_2 - 3) = \\ &\quad \frac{1}{2} d_1(d_1 - d_2)(d_1 + d_2 - 2)^2 + \frac{1}{2} (-2d_1 + d_2 + d)(d_1 + d_2 - 2). \end{aligned}$$

$\square$

We can now prove the following:

**Theorem 4.8.** *There is a Zariski open, dense subset  $U \subset \Omega_2(d_1, d_2)$  such that for every mapping  $F \in U$  the discriminant  $\Delta(F) = F(C(F))$  has only cusps and nodes as singularities. Let  $d = \gcd(d_1, d_2)$ . Then the number of cusps is equal to*

$$c(F) = d_1^2 + d_2^2 + 3d_1 d_2 - 6d_1 - 6d_2 + 7$$

*and the number of nodes is equal to*

$$d(F) = \frac{1}{2} [(d_1 d_2 - 4)((d_1 + d_2 - 2)^2 - 2) - (d - 5)(d_1 + d_2 - 2) - 6].$$

*Proof.* Let  $d_1 \geq d_2$  and  $D = d_1 + d_2 - 2$ . By Lemma 4.3 we have  $\deg \Delta(F) = d_1 D$ . From Lemma 4.1 we know that  $\Delta(F)$  has only cusps and nodes as singularities and is birational with  $C(F)$ . Hence  $\Delta(F)$  has genus  $g = \frac{1}{2}(D-1)(D-2)$ . Thus by Theorem 4.2 we have

$$\frac{1}{2}(d_1 D - 1)(d_1 D - 2) = \frac{1}{2}(D-1)(D-2) + c(F) + d(F) + \sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z.$$

Substituting

$$\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = \frac{1}{2}d_1(d_1 - d_2)D^2 + \frac{1}{2}(-2d_1 + d_2 + d)D$$

from Theorem 4.5 we obtain

$$2(c(F) + d(F)) = d_1 d_2 D^2 - D^2 + 3D - d_1 D - d_2 D - dD = (d_1 d_2 - 2)D^2 - (d-1)D.$$

Thus by Theorem 3.9 we get:

$$\begin{aligned} d(F) &= \frac{1}{2} [(d_1 d_2 - 2)D^2 - (d-1)D - 2(D^2 - 2D + d_1 d_2 - 1)] = \\ &= \frac{1}{2} [(d_1 d_2 - 4)(D^2 - 2) - (d-5)D - 6]. \end{aligned}$$

□

**Remark 4.9.** If  $d_1 = d_2 = d$  then the discriminant has  $2d - 2$  smooth points at infinity and in each of these points it is tangent to the line  $L_\infty$  (at infinity) with multiplicity  $d$ . If  $d_1 > d_2$ , then the discriminant has only one point at infinity with  $d_1 + d_2 - 2$  branches  $V_1, \dots, V_{d_1+d_2-2}$  and each of these branches has delta invariant

$$\delta(V_i) = \frac{(d_1 - 1)(d_1 - d_2 - 1) + (d - 1)}{2}$$

and  $V_i \cdot L_\infty = d_1$ . Additionally  $V_i \cdot V_j = d_1(d_1 - d_2)$ . In particular branches  $V_i$  are smooth if and only if  $d_1 = d_2$  or  $d_1 = d_2 + 1$ .

## 5. THE COMPLEX SPHERE

In the next two sections we show that our method can be easily generalized to the case when  $X$  is a complex sphere. Let  $\phi = y^2 + 2xz$  and let  $S$  be a complex sphere:  $S = \{(x, y, z) : \phi = 1\}$  (of course  $S$  is linearly equivalent with a standard sphere  $S' := \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ ). Here we will study the set  $\Omega_S(d_1, d_2)$ . First we compute the critical set  $C(F)$  of a generic mapping  $F = (f, g) \in \Omega_S(d_1, d_2)$ . Note that  $x \in C(F)$  if  $\text{rank}(\nabla \phi, \nabla f, \nabla g) < 3$ , hence  $C(F)$  is the intersection of  $S$  and the surface given by

$$J(F) = \begin{vmatrix} z & y & x \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = 0.$$

In particular we have:

**Corollary 5.1.** *For a generic mapping  $F \in \Omega_S(d_1, d_2)$  we have  $\deg C(F) = 2(d_1 + d_2 - 1)$ .*

Now we describe cusps of a generic mapping  $F : S \rightarrow \mathbb{C}^2$ . Note that a tangent line to  $C(F)$  is given by two equations:

$$zv_1 + yv_2 + xv_3 = 0, \quad J(F)_x v_1 + J(F)_y v_2 + J(F)_z v_3 = 0.$$



The mapping  $F$  has a cusp in a point  $(x, y, z)$  if

- (1)  $(x, y, z) \in C(F)$
- (2) the line given by the kernel of  $d_{(x,y,z)}F$  is tangent to  $C(F)$ .

First let us determine the kernel of  $d_{(x,y,z)}F$ . If  $\text{rank} \begin{vmatrix} z & y & x \\ f_x & f_y & f_z \end{vmatrix} = 2$ , then the kernel is given by a vector

$$v(f) = \left( \begin{vmatrix} y & x \\ f_y & f_z \end{vmatrix}, -\begin{vmatrix} z & x \\ f_x & f_z \end{vmatrix}, \begin{vmatrix} z & y \\ f_x & f_y \end{vmatrix} \right).$$

Otherwise it is a vector

$$v(g) = \left( \begin{vmatrix} y & x \\ g_y & g_z \end{vmatrix}, -\begin{vmatrix} z & x \\ g_x & g_z \end{vmatrix}, \begin{vmatrix} z & y \\ g_x & g_y \end{vmatrix} \right).$$

Let  $J_{1,1}(F) := J(F)_x v_1(f) + J(F)_y v_2(f) + J(F)_z v_3(f)$  and  $J_{1,2}(F) := J(F)_x v_1(g) + J(F)_y v_2(g) + J(F)_z v_3(g)$ . Let  $C$  denote the set of cusps of  $F$ , for generic  $F$  we have from the construction:

$$C = \{J(F) = J_{1,1}(F) = J_{1,2}(F) = 0\}.$$

Furthermore, we will show in Lemma 5.2 that  $S \cap \{J_{1,2}(F) = 0\} \cap \{v(f) = 0\} = \emptyset$  which gives

$$C = S \cap (\{J(F) = J_{1,1}(F) = 0\} \setminus \{v(f) = 0\}).$$

**Lemma 5.2.** *Let  $L_\infty$  denote the plane at infinity of  $\mathbb{C}^3$ . There is a non-empty open subset  $V \subset \Omega_S(d_1, d_2)$  such that for all  $F = (f, g) \in V$ :*

- (1)  $S \cap \{J_{1,2}(F) = 0\} \cap \{v(f) = 0\} = \emptyset$ ,
- (2)  $\overline{S} \cap \overline{\{J(F) = 0\}} \cap \overline{\{J_{1,1}(F) = 0\}} \cap L_\infty = \emptyset$ ,
- (3)  $\overline{S} \cap \overline{\{J(F) = 0\}} \cap L_\infty$ .

*Proof.* (1) The assertion can be proved locally. Consider the open set  $U_z = \{p \in S : z \neq 0\}$  (and similarly open sets  $U_x, U_y$ ). In  $U_z$  we have globally defined local coordinates  $x, y$ . Now the proof reduces to Lemma 3.7.

(2) Since the case  $d_1 = d_2 = 1$  is trivial we may assume that  $d_1 > 1$  or  $d_2 > 1$ . Similarly as in Lemma 3.5 we will show that there is an open subset  $V \subset H_{d_1, d_2} := H_{d_1} \times H_{d_2}$  such that  $\overline{S} \cap \overline{J(F)} \cap \overline{J_{1,1}(F)} \cap L_\infty = \emptyset$  for all  $F = (f, g) \in V$ . Let  $\phi(x, y, z) = y^2 + 2xz$  and  $\Gamma := \{(x, y, z) \in \mathbb{P}^2 : \phi(x, y, z) = 0\}$ . Obviously  $\Gamma \cong \mathbb{P}^1$ .

Consider the set  $X = \{(p, F) \in \Gamma \times H_{d_1, d_2} : \phi(p) = J(F)(p) = J_{1,1}(F)(p) = 0\}$ . If  $\overline{\phi} \cap \overline{J(F)} \cap \overline{J_{1,1}(F)} \cap L_\infty \neq \emptyset$  then  $F$  belongs to the image of the projection of  $X$  on  $H_{d_1, d_2}$ . So to prove (1) it is sufficient to show that  $X$  has dimension strictly smaller than the dimension of  $H_{d_1, d_2}$ .

Let  $q = (1 : 0 : 0) \in \mathbb{P}^2$ ,  $Y := \{q\} \times H_{d_1, d_2}$  and  $X_0 = X \cap Y$ . Note that all fibers of the projection  $X \rightarrow \Gamma$  are isomorphic to  $X_0$ , because the group  $GL(S)$  of linear transformations of  $S$  acts transitively on the conic at infinity of  $S$ . Thus  $\dim(X) = \dim(X_0) + \dim(\Gamma)$  and to prove (1) it is sufficient to show that  $X_0$  has codimension at least 2 in  $Y$ .

Let  $p = (q, F) \in Y$  and let  $a_{i,j}$  and  $b_{i,j}$  be the parameters in  $H_{d_1, d_2}$  giving respectively the coefficients of  $f$  at  $x^{d_1-i-j}y^i z^j$  and of  $g$  at  $x^{d_2-i-j}y^i z^j$ . For  $0 \leq i + j + k \leq d_1$ , we have  $\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}(q) = \frac{(d_1-j-k)!j!k!}{(d_1-i-j-k)!} a_{j,k}(F)$  and similarly for  $g$  and  $b_{j,k}$ .

To conclude the proof of (1) we will show that the codimension of  $\{a_{1,0}b_{1,0} = 0\} \cap X_0$  in  $Y$  is at least 2 and  $\nabla J$  and  $\nabla J_{1,1}$  are linearly independent outside  $\{a_{1,0}b_{1,0} = 0\} \cap X_0$  and thus the variety  $X_0$  has codimension 2 in  $Y$ .

Let us calculate  $J(p)$ . We have  $J(p) = (f_x g_y - f_y g_x)(q, F) = (d_1 a_{0,0} b_{1,0} - d_2 a_{1,0} b_{0,0})(F)$ . Thus  $\{a_{0,0} = 0\} \cap X_0 \subset \{a_{0,0} = a_{1,0} b_{0,0} = 0\} \cap Y$  has codimension at least 2 and in further calculations we may assume that  $a_{0,0}(F) \neq 0$  and similarly  $b_{0,0}(F) \neq 0$ .

Let us assume that  $d_2 > 1$ , we have  $\frac{\partial J}{\partial b_{1,0}}(p) = \frac{\partial d_1 a_{0,0} b_{1,0} - d_2 a_{1,0} b_{0,0}}{\partial b_{1,0}}(F) = d_1 a_{0,0}(F)$  and  $\frac{\partial J(p)}{\partial b_{2,0}} = 0$ . Now let us calculate  $\frac{\partial J_{1,1}}{\partial b_{2,0}}(p)$ . The coefficient  $b_{2,0}$  can only be obtained from  $\frac{\partial^2 g}{\partial y^2}$ , which is present in  $J_{1,1}$  in the summand  $\frac{\partial^2 g}{\partial y^2} \begin{vmatrix} z & x \\ f_x & f_z \end{vmatrix}^2$ . Thus  $\frac{\partial J_{1,1}}{\partial b_{2,0}}(p) = \frac{\partial(2b_{2,0}d_1^2 a_{0,0}^2)}{\partial b_{2,0}}(F) = 2d_1^2 a_{0,0}(F)^2$ . So  $\det \frac{\partial(J, J_{1,1})}{\partial(b_{0,1}, b_{0,2})}(p) = 2d_1^3 (a_{1,0}(F))^3 \neq 0$ .

Similarly, if  $d_2 = 1$  and  $d_1 > 1$  then  $\det \frac{\partial(J, J_{1,1})}{\partial(a_{0,1}, a_{0,2})}(p) = 2d_1 d_2^2 (a_{1,0}(F))(b_{1,0}(F))^2 \neq 0$ .

(3) Note that  $\overline{\{\nabla J(F)|_S = 0\}} \subset \overline{J_{1,1}(F)}$ , hence (2) implies (3). Indeed in that case  $\nabla J(F)$  is proportional to  $\nabla \phi$ .  $\square$

**Lemma 5.3.** *There is a non-empty open subset  $V_1 \subset \Omega_S(d_1, d_2)$  such that for all  $(f, g) \in V_3$  the curve  $S \cap J(f, g)$  is transversal to the curve  $S \cap J_{1,1}(f, g)$ .*

*Proof.* As in Lemma 5.2 (1) we consider the sets  $U_x, U_y, U_z$  with globally defined local coordinates and reduce the proof to Lemmas 3.6 and Lemma 3.8.  $\square$

**Lemma 5.4.** *There is a non-empty open subset  $V \subset H_{d_1}$  such that for all  $f \in V$  the equations:*

- (1)  $\phi(x, y, z) = 0$ ,
- (2)  $v(f) = 0$

*have no common solutions different from  $(0, 0, 0)$ .*

*Proof.* We proceed similarly as in Lemma 5.2 (2).

Let  $\Gamma := \{(x, y, z) \in \mathbb{P}^2 : \phi(x, y, z) = 0\} \cong \mathbb{P}^1$ . Consider the set

$$X = \{(p, f) \in \Gamma \times H_{d_1} : \phi(p) = v_1(f)(p) = v_2(f)(p) = v_3(f)(p) = 0\}.$$

If  $\{\phi = 0\} \cap \{v(f) = 0\} \neq \emptyset$  then  $f$  belongs to the image of the projection of  $X$  on  $H_{d_1}$ . So to prove (1) it is sufficient to show that  $X$  has dimension strictly smaller than the dimension of  $H_{d_1}$ .

Let  $q = (1 : 0 : 0) \in \mathbb{P}^2$ ,  $Y := \{q\} \times H_{d_1}$  and  $X_0 = X \cap Y$ . As before, all fibers of the projection  $X \rightarrow \Gamma$  are isomorphic to  $X_0$ , so  $\dim(X) = \dim(X_0) + \dim(\Gamma)$  and it is sufficient to show that  $X_0$  has codimension at least 2 in  $Y$ .

But  $X_0$  is given by two equations:  $a_{(1,0)} = 0, -d_1 a_{(0,0)} = 0$ , so  $\text{codim } X_0 = 2$ .  $\square$

**Lemma 5.5.** *There is a non-empty open subset  $V_2 \subset \Omega_S(d_1, d_2)$  such that for all  $(f, g) \in V_2$  the equations:*

- (1)  $y^2 + 2xz = 1$ ,
- (2)  $v(f) = 0$

*have exactly  $2(d_1^2 - d_1 + 1)$  common solutions.*

*Proof.* We have

$$v(f) = \left( \left| \begin{smallmatrix} y & x \\ f_y & f_z \end{smallmatrix} \right|, - \left| \begin{smallmatrix} z & x \\ f_x & f_z \end{smallmatrix} \right|, \left| \begin{smallmatrix} z & y \\ f_x & f_y \end{smallmatrix} \right| \right).$$

Note that generically the curve  $\left\{ \left| \begin{smallmatrix} y & x \\ f_y & f_z \end{smallmatrix} \right| = 0 \right\} \cap \left\{ \left| \begin{smallmatrix} z & x \\ f_x & f_z \end{smallmatrix} \right| = 0 \right\}$  decomposes into  $\{v(f) = 0\}$  and  $\{x = f_z = 0\}$ . Thus by the Bezout Theorem  $\deg\{v(f) = 0\} = d_1^2 - d_1 + 1$  and  $S \cap \{v(f) = 0\}$  has  $2(d_1^2 - d_1 + 1)$  points. We leave checking that the intersections are transversal and there are no components at infinity to the reader.  $\square$

Now we are in a position to prove:

**Theorem 5.6.** *There is a Zariski open, dense subset  $U \subset \Omega_S(d_1, d_2)$  such that for every mapping  $F \in U$  the mapping  $F$  has only folds and cusps as singularities and the number of cusps is equal to*

$$2(d_1^2 + d_2^2 + 3d_1d_2 - 3d_1 - 3d_2 + 1).$$

*Moreover the set  $C(F)$  of critical points of  $F$  is a smooth connected curve, which is topologically equivalent to a sphere with  $g = (d_1 + d_2 - 2)^2$  handles and  $2(d_1 + d_2 - 1)$  points removed.*

*Proof.* First assume that  $d_1, d_2 \geq 2$ . Note that every point  $a$  of the intersection of curves  $J(f, g)$  and  $J_{1,1}(f, g)$  with  $v(f) \neq 0$  is a cusp. Moreover for a generic mapping  $F$  points with  $v(f) = 0$  are not cusps (Lemma 5.2). By Lemma 5.5 we have that in the set  $S \cap \{v(f) = 0\}$  there are exactly  $2(d_1^2 - d_1 + 1)$  points and that the number of cusps of a generic mapping is equal to

$$2[(d_1 + d_2 - 1)(2d_1 + d_2 - 2) - (d_1^2 - d_1 + 1)] = 2(d_1^2 + d_2^2 + 3d_1d_2 - 3d_1 - 3d_2 + 1).$$

Moreover by Lemma 5.2 we have that  $C(F) = S_1(F)$  is a smooth affine curve which is transversal to the plane at infinity. This means that  $J := \overline{C(F)}$  is also smooth at infinity, hence it is a smooth projective curve of degree  $2(d_1 + d_2 - 1)$ . Note that  $\text{Pic}(\overline{S}) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$ , where  $L_1, L_2$  are suitable lines in  $\overline{S}$ . Moreover if  $H$  is a plane section then  $H \sim L_1 + L_2$ . Hence in  $\text{Pic}(\overline{S})$  we have  $\overline{C(F)} \sim aL_1 + bL_2$  where  $a + b = 2(d_1 + d_2 - 1)$ .

Take  $l_i = L_i \cap S$  and note that  $\text{Pic}(S)$  is generated freely by  $l_1$  or  $l_2$  with the relation  $l_1 + l_2 = 0$ . In particular  $C(F) \sim (a - b)l_1$ . But in  $\text{Pic}(S)$  we have  $C(F) \sim (d_1 + d_2 - 1)H = 0$ . Thus  $a = b = d_1 + d_2 - 1$ .

Suppose that  $C(F)$  is not connected. Hence  $\overline{C(F)} = \Gamma_1 + \Gamma_2$ . We have  $\Gamma_1 \sim a_1L_1 + b_1L_2$  and  $\Gamma_2 \sim a_2L_1 + b_2L_2$ , where  $a_1, b_1, a_2, b_2 \geq 0$ ,  $a_1 + b_1 > 0$  and  $a_2 + b_2 > 0$ . Note that  $a_1 + a_2 = b_1 + b_2 = d_1 + d_2 - 1 > 0$  thus if  $a_1b_2 = 0$  then  $a_2b_1 > 0$ . So  $\Gamma_1 \cdot \Gamma_2 = a_1b_2 + a_2b_1 > 0$ . Consequently  $\Gamma_1 \cap \Gamma_2 \neq \emptyset$  and  $\overline{C(F)}$  is not smooth – a contradiction. This implies that  $C(F)$  is connected.

Let  $H \subset \mathbb{P}^3$  be a hyperplane. The canonical divisor of  $\overline{S}$  is  $-2H = -2(L_1 + L_2)$ . Hence  $K_J = (J - 2H)|_J = (d_1 + d_2 - 3)(L_1 + L_2)|_J$  and  $\deg K_J = 2(d_1 + d_2 - 3)(d_1 + d_2 - 1)$ . By Riemann-Roch Theorem  $J$  has genus  $g = \deg K_J / 2 + 1 = (d_1 + d_2 - 2)^2$ . This means in particular that  $\overline{C(F)}$  is homeomorphic to a sphere with  $g = (d_1 + d_2 - 2)^2$  handles. Moreover, by the Bezout Theorem it has precisely  $2(d_1 + d_2 - 1)$  points at infinity.

Now go back to the case when  $d_1 = 1$  or  $d_2 = 1$ . Since the case  $d_1 = d_2 = 1$  is trivial we can assume that  $d_1 > d_2 = 1$ . Consider the mapping  $\Psi : \Omega_S(d_1, 1) \times S \ni (F, x) \rightarrow j^2(F)(x) \in J^2(X, \mathbb{C}^2)$ . We show that:

- 1)  $Y := \Psi(\Omega_S(d_1, 1) \times S)$  is a smooth subvariety of  $J^2(X, \mathbb{C}^2)$ .

2)  $S_{1,1} \pitchfork Y$ .

3) The mapping  $\Psi : \Omega_S(d_1, 1) \times S \rightarrow Y$  is a submersion.

Take  $(f, g) \in \Omega_S(d_1, 1)$ . Thus  $g = a + bx + cy + dz$ . By the symmetry we can restrict only to the set  $U_z = \{(x, y, z) \in S : z \neq 0\}$ . By direct computation we have that the rank  $\Psi$  is constant and equal to  $\dim J^2(X, \mathbb{C}^2) - 2$ . Moreover,

$$\frac{\partial z^2}{\partial x^2} = \frac{y^2 - 1}{z^3}, \frac{\partial z^2}{\partial y^2} = \frac{x^2 - 1}{z^3}, \frac{\partial z^2}{\partial xy} = \frac{-xy}{z^3}.$$

Since  $\frac{\partial g^2}{\partial x^2}$  is proportional to  $\frac{\partial z^2}{\partial x^2}$  etc., we have

- (1)  $\frac{\partial g^2}{\partial x^2}(x^2 - 1) - (y^2 - 1)\frac{\partial g^2}{\partial y^2} = 0$ ,
- (2)  $\frac{\partial g^2}{\partial x^2}xy + (y^2 - 1)\frac{\partial g^2}{\partial xy} = 0$ ,
- (3)  $\frac{\partial g^2}{\partial y^2}xy + (x^2 - 1)\frac{\partial g^2}{\partial xy} = 0$ .

Considering equations (1) and (2) we see that  $\Psi(\Omega_S(d_1, 1) \times U_z)$  is smooth outside  $y = \pm 1$ , considering equations (1) and (3) we see that  $\Psi(\Omega_S(d_1, 1) \times U_z)$  is smooth outside  $x = \pm 1$ . Considering equations (2) and (3) we see that  $\Psi(\Omega_S(d_1, 1) \times S)$  is smooth outside  $xy = 0$ . Consequently the image  $\Psi(\Omega_S(d_1, 1) \times U_z)$  is smooth (we differentiate these equation with respect to  $g_{xx}, g_{xy}, g_{yy}$ ). Since the mapping  $\Psi$  has constant rank it is a submersion onto its image.

Moreover, using the same equations it is easy to check that  $S_{1,1}$  is transversal to  $Y$ . Indeed we use the equations (1), (2) (3) for  $Y$  and standard equation for  $S_{1,1}$  (see section 3). We differentiate equation for  $Y$  with respect to  $x, y$  and equation for  $S_{1,1}$  with respect to  $f_x, f_y, f_{xx}, f_{yy}, g_x, g_y, g_{xx}, g_{yy}$ . In particular we get that  $S_{1,1} \pitchfork \Psi(\Omega_S(d_1, 1) \times U_z)$  for  $z \neq \pm 1$ . Similarly  $S_{1,1} \pitchfork \Psi(\Omega_S(d_1, 1) \times U_x)$  for  $x \neq \pm 1$  and  $S_{1,1} \pitchfork \Psi(\Omega_S(d_1, 1) \times U_y)$  for  $y \neq \pm 1$ . Thus  $S_{1,1} \pitchfork Y$ . In the same way we can check that other conditions of our proof are satisfied.  $\square$

## 6. THE COMPLEX SPHERE: THE DISCRIMINANT

Here we analyze the discriminant of a generic mapping from  $\Omega_S(d_1, d_2)$ . Let us recall that the discriminant of the mapping  $F : S \rightarrow \mathbb{C}^2$  is the curve  $\Delta(F) := F(C(F))$ , where  $C(F)$  is the critical curve of  $F$ . We have:

**Lemma 6.1.** *There is a non-empty open subset  $U \subset \Omega_S(d_1, d_2)$  such that for every mapping  $F \in U$ :*

- (1)  $F|_{C(F)}$  is injective outside a finite set,
- (2) if  $p \in \Delta(F)$  then  $|F^{-1}(p) \cap C(F)| \leq 2$ ,
- (3) if  $|F^{-1}(p) \cap C(F)| = 2$  then the curve  $\Delta(F)$  has a normal crossing at  $p$ .

*Proof.* For this proof we will assume that  $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ . We may assume that  $d_1 \geq d_2$  and since the assertion is trivial for  $d_1 = d_2 = 1$  we may also assume that  $d_1 > 1$ .

To prove (1) consider the set  $X = \{(p, q, F) \in S \times S \times \Omega_S(d_1, d_2) : p \neq q, F(p) = F(q), J(F)(p) = J(F)(q) = 0\}$ . We will show that  $X$  has dimension not greater than  $\dim \Omega_S(d_1, d_2)$ . So the projection of  $X$  on  $\Omega_S(d_1, d_2)$  has finite fibers on some open subset  $U \subset \Omega_S(d_1, d_2)$ . Moreover if the fiber over  $F$  is finite then  $F|_{C(F)}$  is injective outside a finite set given by the fiber.

Let  $p = (0, 0, 1)$ ,  $q = (0, \alpha, \beta) \in S \setminus p$ ,  $Y_q := \{p\} \times \{q\} \times \Omega_S(d_1, d_2)$  and  $X_q = X \cap Y_q$ . Note that for every pair  $(p', q') \in S^2$  there is a rotation  $T$  such that  $T(p') = p$  and  $T(q') = q$ . Moreover  $(p, q, F) \mapsto (p', q', F \circ T)$  is an isomorphism of the fiber over  $(p', q')$  of the projection  $X \rightarrow S \times S$  with  $X_q$ . Thus to prove (1) it is sufficient to show that every  $X_q$  has codimension at least 4 in  $Y_q$ .

Let  $a_{ij}, b_{ij}, c_{ij}$  and  $d_{ij}$  be such parameters in  $\Omega_2(d_1, d_2)$  that  $F(x, y, z) = (\sum a_{ij} x^i y^j + z \sum b_{ij} x^i y^j, \sum c_{ij} x^i y^j + z \sum d_{ij} x^i y^j)$ .

The condition  $F(p) = F(q)$  yields the equations  $w_1 = \sum_{j=1}^{d_1} a_{0j} \alpha^j + (\beta - 1)b_{00} + \beta \sum_{j=1}^{d_1-1} b_{0j} \alpha^j = 0$  and  $w_2 = \sum_{j=1}^{d_1} c_{0j} \alpha^j + (\beta - 1)d_{00} + \beta \sum_{j=1}^{d_2-1} d_{0j} \alpha^j = 0$ . The condition  $J(F)(p) = 0$  gives  $w_3 = (a_{10} + b_{10})(c_{01} + d_{01}) - (a_{01} + b_{01})(c_{10} + d_{10}) = 0$  and the condition  $J(F)(q) = 0$  gives  $w_4 = w_{4,1}w_{4,2} - w_{4,3}w_{4,4}$ , where  $w_{4,1} = \sum_{j=0}^{d_1-1} a_{1j} \alpha^j + \beta \sum_{j=0}^{d_1-2} b_{1j} \alpha^j$ ,  $w_{4,2} = \beta \sum_{j=1}^{d_2} j c_{0j} \alpha^{j-1} + \beta^2 \sum_{j=1}^{d_2-1} j d_{0j} \alpha^{j-1} - \alpha \sum_{j=0}^{d_2-1} d_{0j} \alpha^j$ ,  $w_{4,3} = \sum_{j=0}^{d_2-1} c_{1j} \alpha^j + \beta \sum_{j=0}^{d_2-2} d_{1j} \alpha^j$  and  $w_{4,4} = \beta \sum_{j=1}^{d_1} j a_{0j} \alpha^{j-1} + \beta^2 \sum_{j=1}^{d_1-1} j b_{0j} \alpha^{j-1} - \alpha \sum_{j=0}^{d_1-1} b_{0j} \alpha^j$ .

Note that the matrix  $\frac{\partial(w_1, w_2, w_3, w_4)}{\partial(b_{00}, d_{00}, a_{10}, a_{11})}$  is triangular and its determinant is equal to  $\alpha(\beta - 1)^2(c_{01} + d_{01})w_{4,2}$ . Thus if  $\alpha \neq 0$  then the equations  $w_1, w_2, w_3, w_4$  are independent outside the set  $\{c_{01} + d_{01} = 0\} \cup \{w_{4,2} = 0\}$ . Moreover  $X_q \cap \{c_{01} + d_{01} = 0\}$  is a subset of  $w_1 = w_2 = c_{01} + d_{01} = 0 \cap (\{a_{01} + b_{01} = 0\} \cup \{c_{10} + d_{10} = 0\})$ , which obviously has codimension 4. Similarly  $X_q \cap \{w_{4,2} = 0\} \subset \{w_1 = w_2 = w_{4,2} = 0\} \cap (\{w_{4,3} = 0\} \cup \{w_{4,4} = 0\})$  has codimension at least 4 because  $\det\left(\frac{\partial(w_1, w_2, w_{4,2}, w_{4,3})}{\partial(b_{00}, d_{00}, c_{01}, c_{10})}\right) = (\beta - 1)(\beta^2 - \beta + \alpha^2) = -(\beta - 1)^2 \neq 0$  and  $\det\left(\frac{\partial(w_1, w_2, w_{4,2}, w_{4,4})}{\partial(b_{00}, d_{00}, c_{01}, a_{01})}\right) = (\beta^2 - \beta + \alpha^2)^2 = (\beta - 1)^2 \neq 0$ .

It remains to check the codimension of  $X_q$  when  $\alpha = 0$ , i.e  $q = (0, 0, -1)$ . In this case we have  $w_1 = b_{00}$ ,  $w_2 = d_{00}$ ,  $w_3 = (a_{10} + b_{10})(c_{01} + d_{01}) - (a_{01} + b_{01})(c_{10} + d_{10})$  and  $w_4 = -(a_{10} - b_{10})(c_{01} - d_{01}) + (a_{01} - b_{01})(c_{10} - d_{10})$ . Similarly as above note that  $\det\left(\frac{\partial(w_1, w_2, w_3, w_4)}{\partial(b_{00}, d_{00}, a_{10}, b_{10})}\right) = 2(c_{01} + d_{01})(c_{01} - d_{01})$  so the equations  $w_1, w_2, w_3, w_4$  are independent outside the set  $\{(c_{01} + d_{01})(c_{01} - d_{01}) = 0\}$ . However  $X_q \cap \{(c_{01} + d_{01})(c_{01} - d_{01}) = 0\}$  has obviously codimension at least 4 in  $Y_q$ . Thus  $\dim X_q \leq \dim Y_q - 4$ , which concludes the proof of (1).

To prove (2) consider the set  $X = \{(p, q, r, F) \in S \times S \times S \times \Omega_S(d_1, d_2) : p \neq q \neq r \neq p, F(p) = F(q) = F(r), J(F)(p) = J(F)(q) = J(F)(r) = 0\}$ . Similarly as in (1) we compute that  $X$  has codimension at least 7. It follows that the projection of  $X$  on  $\Omega_2(d_1, d_2)$  has empty fibers on some open subset  $U \subset \Omega_2(d_1, d_2)$ . The computation is purely technical and similar to the computation in (1), so we leave the details to the reader.

To prove (3) note that if  $q \in C(F)$  then  $d_q F(T_q C(F))$  is spanned by the vector  $(J_{1,1}(F)(q), J_{1,2}(F)(q))$ . Thus if  $F^{-1}(p) \cap C(F) = \{q_1, q_2\}$  then  $\Delta(F)$  has a normal crossing at  $p$  if and only if  $(J_{1,1}(F)(q_1), J_{1,2}(F)(q_1))$  and  $(J_{1,1}(F)(q_2), J_{1,2}(F)(q_2))$  are independent, i.e.  $J_{1,1}(F)(q_1)J_{1,2}(F)(q_2) - J_{1,2}(F)(q_1)J_{1,1}(F)(q_2) \neq 0$ . Similarly as in (1) let us consider the set  $X = \{(p, q, F) \in S \times S \times \Omega_S(d_1, d_2) : p \neq q, F(p) = F(q), J(F)(p) = J(F)(q) = J_{1,1}(F)(p)J_{1,2}(F)(q) - J_{1,2}(F)(p)J_{1,1}(F)(q) = 0\}$ . One can compute that  $X$  has codimension at least 5, thus the projection of  $X$  on  $\Omega_S(d_1, d_2)$  has empty fibers on some open subset  $U \subset \Omega_S(d_1, d_2)$ .  $\square$

Hence for a generic  $F$  the only singularities of  $\Delta(F)$  are cusps and nodes. We showed in Theorem 5.6 that there are exactly  $c(F) = 2(d_1^2 + d_2^2 + 3d_1d_2 - 3d_1 - 3d_2 + 1)$  cusps. Now we will compute the number  $d(F)$  of nodes of  $\Delta(F)$ . First we compute the degree of the discriminant:

**Lemma 6.2.** *Let  $F = (f, g) \in \Omega_S(d_1, d_2)$  be a generic mapping. If  $d_1 \geq d_2$  then  $\deg \Delta(F) = 2d_1(d_1 + d_2 - 1)$ .*

*Proof.* Let  $L \subset \mathbb{C}^2$  be a generic line  $\{ax + by + c = 0\}$ . Then  $L$  intersects  $\Delta(F)$  in smooth points and  $\deg \Delta(F) = \#L \cap \Delta(F)$ . If  $j : C(F) \rightarrow \Delta(F)$  is a mapping induced by  $F$  then  $\#L \cap \Delta(F) = \#j^{-1}(L \cap \Delta(F))$ . The curve  $j^{-1}(L) = \{af + bg + c = 0\}$  has no common points at infinity with  $C(F)$ . Hence by Bezout Theorem we have  $\#j^{-1}(L \cap \Delta(F)) = (\deg j^{-1}(L))(\deg C(F)) = 2d_1(d_1 + d_2 - 1)$ . Consequently  $\deg \Delta(F) = 2d_1(d_1 + d_2 - 1)$ .  $\square$

The main result of this section will be based on the following:

**Theorem 6.3.** *Let  $F \in \Omega_S(d_1, d_2)$  be a generic mapping. Let  $d_1 \geq d_2$  and  $d = \gcd(d_1, d_2)$ . Denote by  $\overline{\Delta}$  the projective closure of the discriminant  $\Delta$ . Then*

$$\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = 2d_1(d_1 - d_2)(d_1 + d_2 - 1)^2 + (-2d_1 + d_2 + d)(d_1 + d_2 - 1).$$

*Proof.* Let  $\tilde{f}(x, y, z, w) = w^{d_1} f(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$  and  $\tilde{g}(x, y, z, w) = w^{d_2} g(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$  be the homogenizations of  $f$  and  $g$  and let  $\overline{f}(x, y, w) = \tilde{f}(x, y, 1, w)$  and  $\overline{g}(x, y, w) = \tilde{g}(x, y, 1, w)$ . For a generic mapping the curves  $C(F)$  and  $\{f = 0\}$  have no common points at infinity (see Lemma 6.4). Moreover since  $F$  is generic, we have  $\{z = 0\} \cap \overline{C(F)} = \emptyset$ . Thus  $F$  extends to a neighborhood of  $\overline{C(F)} \cap L_\infty$  on which it is given by the formula

$$\overline{F}(x, y, w) = \left( w^{d_1 - d_2} \frac{\overline{g}(x, y, w)}{\overline{f}(x, y, w)}, \frac{w^{d_1}}{\overline{f}(x, y, w)} \right).$$

Let  $\Gamma = \overline{S} \cap L_\infty$  and  $\Gamma^* = \Gamma \setminus \{(1 : 0 : 0 : 0)\}$ . Let  $\{P_1, \dots, P_{2d_1 + 2d_2 - 2}\} = \overline{C(F)} \cap \Gamma^*$ , fix a point  $P = P_i$ . The curve  $\overline{C(F)}$  is transversal to the line at infinity so it has a local parametrization at  $P$  of the form  $\gamma(t) := (\sum_i a_i t^i, \sum_i b_i t^i, t)$ . We have the following:

**Lemma 6.4.** *If  $F$  is a generic mapping then  $\overline{f}(P) \neq 0$ ,  $\overline{g}(P) \neq 0$  and*

$$\overline{f}(\gamma(t)) = \overline{f}(P)(1 + ct + \dots), \quad \overline{g}(\gamma(t)) = \overline{g}(P)(1 + dt + \dots),$$

where  $cd \neq 0$  and  $d_2c \neq d_1d$ .

*Proof.* Let  $\tilde{J}$  be the homogenization of  $J$ . Obviously

$$\tilde{J}(F) = \begin{vmatrix} z & y & x \\ \tilde{f}_x & \tilde{f}_y & \tilde{f}_z \\ \tilde{g}_x & \tilde{g}_y & \tilde{g}_z \end{vmatrix}.$$

Now let  $\overline{J}(x, y, w) = \tilde{J}(x, y, 1, w)$  and  $\psi(x, y, w) = x + y^2 - w^2 = \tilde{\phi}(x, y, 1, w)$ , where  $\tilde{\phi}$  is the homogenization of  $\phi$ . We have  $\overline{J}(\gamma(t)) = 0$  and  $\psi(\gamma(t)) = 0$ . Moreover,  $\frac{\partial \gamma(t)}{\partial t}|_{t=0} = (a_1, b_1, 1)$ . Thus we have

$$\begin{aligned} \frac{\partial \psi}{\partial x}(P)a_1 + \frac{\partial \psi}{\partial y}(P)b_1 + \frac{\partial \psi}{\partial w}(P) &= 0, \\ \frac{\partial \overline{J}}{\partial x}(P)a_1 + \frac{\partial \overline{J}}{\partial y}(P)b_1 + \frac{\partial \overline{J}}{\partial w}(P) &= 0. \end{aligned}$$

Consequently  $a_1 = \overline{a_1}\delta^{-1}$  and  $b_1 = \overline{b_1}\delta^{-1}$ , where



$$\begin{aligned}\overline{a_1} &= \frac{\partial\psi(P)}{\partial w} \frac{\partial\overline{J}(P)}{\partial y} - \frac{\partial\psi(P)}{\partial y} \frac{\partial\overline{J}(P)}{\partial w}, \quad \overline{b_1} = \frac{\partial\psi(P)}{\partial x} \frac{\partial\overline{J}(P)}{\partial w} - \frac{\partial\psi(P)}{\partial w} \frac{\partial\overline{J}(P)}{\partial x}, \\ \delta &= \frac{\partial\psi(P)}{\partial x} \frac{\partial\overline{J}(P)}{\partial y} - \frac{\partial\psi(P)}{\partial y} \frac{\partial\overline{J}(P)}{\partial x}.\end{aligned}$$

Thus

$$\overline{f}(P)c\delta = \frac{\partial\overline{f}(P)}{\partial x}\overline{a_1} + \frac{\partial\overline{f}(P)}{\partial y}\overline{b_1} + \frac{\partial\overline{f}(P)}{\partial w}\delta.$$

Take

$$\begin{aligned}\tilde{a}_1 &= \frac{\partial\tilde{\psi}(P)}{\partial w} \frac{\partial\tilde{J}(P)}{\partial y} - \frac{\partial\tilde{\psi}(P)}{\partial y} \frac{\partial\tilde{J}(P)}{\partial w}, \quad \tilde{b}_1 = \frac{\partial\tilde{\psi}(P)}{\partial x} \frac{\partial\tilde{J}(P)}{\partial w} - \frac{\partial\tilde{\psi}(P)}{\partial w} \frac{\partial\tilde{J}(P)}{\partial x}, \\ \tilde{\delta} &= \frac{\partial\tilde{\psi}(P)}{\partial x} \frac{\partial\tilde{J}(P)}{\partial y} - \frac{\partial\tilde{\psi}(P)}{\partial y} \frac{\partial\tilde{J}(P)}{\partial x}.\end{aligned}$$

Consider the set

$$X = \left\{ (p, F) \in \Gamma^* \times \Omega_S(d_1, d_2) : \tilde{J}(F)(p) = \frac{\partial\tilde{f}(p)}{\partial x}\tilde{a}_1 + \frac{\partial\tilde{f}(p)}{\partial y}\tilde{b}_1 + \frac{\partial\tilde{f}(p)}{\partial w}\tilde{\delta} = 0 \right\}.$$

Note that if  $\overline{f}(P) = 0$  or  $c = 0$  then the fiber over  $P$  of the projection from  $X$  to  $\Omega_S(d_1, d_2)$  is non-empty. Hence it suffices to prove that  $X$  has codimension at least 2.

Let  $p = (0 : 0 : 1 : 0)$ , and  $q = (-a^2 : a : 1 : 0) \in \Gamma^*$ . Let  $T(x, y, z, w) = (x + 2ay - a^2z, y - az, z, w)$  so  $T(S) = S$  and  $T(q) = p$ . As in Lemma 4.6 we can show that  $(p, F) \mapsto (T^{-1}(p), F \circ T)$  is an isomorphism of  $X_p := X \cap (\{p\} \times \Omega_S(d_1, d_2))$  and  $X \cap (\{q\} \times \Omega_S(d_1, d_2))$ . So it is enough to show that  $X_p$  has codimension 2 in  $Y_p := \{p\} \times \Omega_S(d_1, d_2)$ .

Let  $a_{i,j,k}$  be the parameters in  $\Omega_S(d_1, d_2)$  giving the coefficients of  $\tilde{f}$  at  $x^i y^j z^{d_1-i-j-k} w^k$  (i.e. of  $f$  at  $x^i y^j z^{d_1-i-j-k}$ ) and let  $b_{i,j,k}$  describe the analogous coefficients of  $\tilde{g}$ .

The first equation of  $X_p$  is  $w_1 := d_2 a_{0,1,0} b_{0,0,0} - d_1 b_{0,1,0} a_{0,0,0}$ . The second one is

$$w_2 = a_{1,0,0} \tilde{a}_1 + a_{0,1,0} \tilde{b}_1 + a_{0,0,1} \tilde{\delta} = a_{0,1,0} \frac{\partial\tilde{J}(p)}{\partial w} + a_{0,0,1} \frac{\partial\tilde{J}(p)}{\partial y} = a_{0,1,0}((d_2 - 1)b_{0,0,1}a_{0,1,0} +$$

$$d_2 b_{0,0,0}a_{0,1,1} - d_1 a_{0,0,0}b_{0,1,1} - (d_1 - 1)a_{0,0,1}b_{0,1,0}) + a_{0,0,1}(d_2 a_{1,0,0}b_{0,0,0} - d_1 a_{0,0,0}b_{1,0,0}).$$

By direct computation we obtain

$$\frac{\partial w_1}{\partial b_{0,0,0}} = d_2 a_{0,1,0}, \quad \frac{\partial w_2}{\partial b_{0,0,1}} = (d_2 - 1)a_{0,1,0}^2$$

, Thus if  $d_2 > 1$  then the equations  $w_1 = 0$  and  $w_2 = 0$  are independent outside the set  $\{a_{0,1,0} = 0\}$ . However,  $\{a_{0,1,0} = 0\} \cap w_1 = 0 \subset \{a_{0,1,0} = b_{0,1,0} = 0\} \cup \{a_{0,1,0} = a_{0,0,0} = 0\}$  so  $X \cap Y$  has codimension 2 in  $Y$ . On the other hand, if  $d_1 > 1$  then considering  $\frac{\partial w_2}{\partial a_{0,1,1}}$  instead of  $\frac{\partial w_2}{\partial b_{0,0,1}}$  leads to the desired result.

Finally note that if  $d_2 c = d_1 d$  then

$$d_2 \overline{g}(P) \left( \frac{\partial\tilde{f}(P)}{\partial x}\tilde{a}_1 + \frac{\partial\tilde{f}(P)}{\partial y}\tilde{b}_1 + \frac{\partial\tilde{f}(P)}{\partial w}\tilde{\delta} \right) = d_1 \overline{f}(P) \left( \frac{\partial\tilde{g}(P)}{\partial x}\tilde{a}_1 + \frac{\partial\tilde{g}(P)}{\partial y}\tilde{b}_1 + \frac{\partial\tilde{g}(P)}{\partial w}\tilde{\delta} \right).$$

Hence we consider the set

$$X = \left\{ (p, F) \in \Gamma \times \Omega_S(d_1, d_2) : \tilde{J}(F)(p) = \right.$$

$$d_2 \bar{g}(P) \left( \frac{\partial \tilde{f}(P)}{\partial x} \tilde{a}_1 + \frac{\partial \tilde{f}(P)}{\partial y} \tilde{b}_1 + \frac{\partial \tilde{f}(P)}{\partial w} \tilde{\delta} \right) = d_1 \bar{f}(P) \left( \frac{\partial \tilde{g}(P)}{\partial x} \tilde{a}_1 + \frac{\partial \tilde{g}(P)}{\partial y} \tilde{b}_1 + \frac{\partial \tilde{g}(P)}{\partial w} \tilde{\delta} \right) = 0 \}.$$

Similarly as above one can show that it has codimension 2, which concludes the proof.  $\square$

Let  $C_P$  be the branch of  $\overline{C(F)}$  at  $P$ . Exactly as in the section 4 we have  $2\delta(C_P)_{\overline{F}(P)} = (d_1 - 1)(d_1 - d_2 - d) + (d_1 + 1 - 1)(d - 1) = (d_1 - 1)(d_1 - d_2 - 1) + (d - 1)$ , where  $d = \gcd(d_1, d_2)$ .

To proceed further we also need:

**Lemma 6.5.** *If  $F$  is a generic mapping then*

$$\bar{f}(P_i)^{d_2} \bar{g}(P_j)^{d_1} \neq \bar{f}(P_j)^{d_2} \bar{g}(P_i)^{d_1}$$

for  $i, j \in \{1, 2, \dots, 2(d_1 + d_2 - 1)\}$  and  $i \neq j$ .

*Proof.* Consider the set  $X = \{(p, q, F) \in \Gamma \times \Gamma \times \Omega_S(d_1, d_2) : p \neq q, \tilde{J}(F)(p) = \tilde{J}(F)(q) = \tilde{f}(p)^{d_2} \tilde{g}(q)^{d_1} - \tilde{f}(q)^{d_2} \tilde{g}(p)^{d_1} = 0\}$ . Similarly as in Lemma 6.4 we will prove that  $X$  has codimension 3, so there is a dense open subset  $U \subset \Omega_S(d_1, d_2)$  such that the projection from  $X$  has empty fibers over  $F \in U$ .

Indeed, take  $p = (1 : 0 : 0 : 0)$ ,  $q = (0 : 0 : 1 : 0)$  and  $Y := \{(p, q)\} \times \Omega_3(d_1, d_2)$ , it suffices to show that  $X_0 = X \cap Y$  has codimension 3 in  $Y$ . Let  $a_{ij}$  and  $b_{ij}$  be the parameters in  $\Omega_3(d_1, d_2)$  giving respectively the coefficients of  $\tilde{f}$  at  $x^{d_1-i-j}y^iz^j$  and of  $\tilde{g}$  at  $x^{d_2-i-j}y^iz^j$ .

The three equations describing  $X_0$  are  $w_1 = d_1 a_{0,0} b_{0,1} - d_2 a_{0,1} b_{0,0} = 0$ ,  $w_2 = d_2 a_{1,d_1-1} b_{0,d_2} - d_1 a_{0,d_1} b_{1,d_2-1} = 0$  and  $w_3 = a_{0,0}^{d_2} b_{0,d_2}^{d_1} - a_{0,d_1}^{d_2} b_{0,0}^{d_1} = 0$ . Note that  $X_0 \cap \{a_{0,0} = 0\} = \{a_{0,0} = b_{0,0} = w_2 = 0\} \cup \{a_{0,0} = a_{0,1} = w_2 = 0\}$  has codimension 3. Similarly  $X_0 \cap \{b_{0,0} = 0\}$  and  $X_0 \cap \{a_{0,d_1} = 0\}$  have codimension 3, however outside the set  $\{a_{0,0} = 0\} \cup \{b_{0,0} = 0\} \cup \{a_{0,d_1} = 0\}$  the three equations are obviously independent. Thus  $X_0$  has codimension 3 in  $X$ .  $\square$

Now we are in a position to compute  $\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z$ . If  $d_1 = d_2$  then  $\overline{\Delta}$  has exactly  $2(d_1 + d_2 - 1)$  smooth points at infinity and consequently  $\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = 0$ . So assume  $d_1 > d_2$ , then  $\overline{\Delta}$  has only one point at infinity  $Q = (1 : 0 : 0)$ . In  $Q$  the curve  $\overline{\Delta}$  has exactly  $r = 2(d_1 + d_2 - 1)$  branches  $V_i = \overline{F}(C_{P_i})$ . We have  $2\delta(V_i)_Q = (d_1 - 1)(d_1 - d_2 - 1) + (d - 1)$ . As in the section 4 we have  $V_i \cdot V_j = d_1(d_1 - d_2)$ . Consequently

$$\begin{aligned} \sum_i \delta(V_i) + \sum_{i>j} V_i \cdot V_j &= [(d_1 - 1)(d_1 - d_2 - 1) + (d - 1)](d_1 + d_2 - 1) + \\ & d_1(d_1 - d_2)(d_1 + d_2 - 1)(2(d_1 + d_2 - 1) - 1) = \\ & 2d_1(d_1 - d_2)(d_1 + d_2 - 1)^2 + (-2d_1 + d_2 + d)(d_1 + d_2 - 1). \end{aligned}$$

Moreover, if  $d_1 = d_2 = d$  then the discriminant has  $4d - 2$  smooth points at infinity and in each of these points it is tangent to the plane  $\pi_\infty$  (at infinity) with multiplicity  $d$ . If  $d_1 > d_2$ , then the discriminant has only one point at infinity with  $2(d_1 + d_2 - 1)$  branches  $V_1, \dots, V_{2(d_1+d_2-1)}$  and each of these branches has delta invariant

$$\delta(V_i) = \frac{(d_1 - 1)(d_1 - d_2 - 1) + (d - 1)}{2}$$

and  $V_i \cdot L_\infty = d_1$ . Additionally  $V_i \cdot V_j = d_1(d_1 - d_2)$ . In particular branches  $V_i$  are smooth if and only if  $d_1 = d_2$  or  $d_1 = d_2 + 1$ .  $\square$

We can now prove the following:

**Theorem 6.6.** *There is a Zariski open, dense subset  $U \subset \Omega_S(d_1, d_2)$  such that for every mapping  $F \in U$  the discriminant  $\Delta(F) = F(C(F))$  has only cusps and nodes as singularities. Then the number of cusps is equal to*

$$c(F) = 2(d_1^2 + d_2^2 + 3d_1d_2 - 3d_1 - 3d_2 + 1)$$

and the number of nodes is equal to

$$d(F) = (2d_1d_2 - 3)D^2 - D(d_1 + d_2 + d - 2) - 2(d_1d_2 - d_1 - d_2),$$

where  $D = d_1 + d_2 - 1$  and  $d = \gcd(d_1, d_2)$ .

*Proof.* Let  $d_1 \geq d_2$  and  $D = (d_1 + d_2 - 1)$ . By Lemma 6.2 we have  $\deg \Delta(F) = 2d_1D$ . From Lemma 4.1 we know that  $\Delta(F)$  has only cusps and nodes as singularities and is birational with  $C(F)$ . Hence  $\Delta(F)$  has genus  $g = D(D - 2) + 1$ . Thus by Theorem 4.2 we have

$$(2d_1D - 1)(2d_1D - 2) = D(D - 2) + 1 + c(F) + d(F) + \sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z.$$

Substituting

$$\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = 2d_1(d_1 - d_2)D^2 + (-2d_1 + d_2 + d)D$$

we obtain

$$c(F) + d(F) = (2d_1d_2 - 1)D^2 - D(d_1 + d_2 + d - 2).$$

Thus by Theorem 3.9 we get:

$$\begin{aligned} d(F) &= (2d_1d_2 - 1)D^2 - D(d_1 + d_2 + d - 2) - 2(d_1^2 + d_2^2 + 3d_1d_2 - 3d_1 - 3d_2 + 1) = \\ &= (2d_1d_2 - 3)D^2 - D(d_1 + d_2 + d - 2) - 2(d_1d_2 - d_1 - d_2). \end{aligned}$$

□

**Remark 6.7.** If  $d_1 = d_2 = d$  then the discriminant has  $4d - 2$  smooth points at infinity and in each of these points it is tangent to the line  $L_\infty$  (at infinity) with multiplicity  $d$ . If  $d_1 > d_2$ , then the discriminant has only one point at infinity with  $2(d_1 + d_2 - 1)$  branches  $V_1, \dots, V_{2(d_1+d_2-1)}$  and each of these branches has delta invariant

$$\delta(V_i) = \frac{(d_1 - 1)(d_1 - d_2 - 1) + (d - 1)}{2}$$

and  $V_i \cdot L_\infty = d_1$ . Additionally  $V_i \cdot V_j = d_1(d_1 - d_2)$ . In particular branches  $V_i$  are smooth if and only if  $d_1 = d_2$  or  $d_1 = d_2 + 1$ .

## 7. GENERALIZED CUSPS

In this section our aim is to estimate the number of cusps of non-generic mappings. We start from:

**Definition 7.1.** Let  $F : (\mathbb{C}^2, a) \rightarrow (\mathbb{C}^2, F(a))$  be a germ of a holomorphic mapping. We say that  $F$  has a generalized cusp at  $a$  if  $F_a$  is proper, the curve  $J(F) = 0$  is reduced near  $a$  and the discriminant of  $F_a$  is not smooth at  $F(a)$ .

**Remark 7.2.** If  $F_a$  is proper,  $J(F) = 0$  is reduced near  $a$  and  $J(F)$  is singular at  $a$  then it follows from Corollary 1.11 from [9] that also the discriminant of  $F_a$  is singular at  $F(a)$  and hence  $F$  has a generalized cusp at  $a$ .

Now we introduce the index of generalized cusp:

**Definition 7.3.** Let  $F = (f, g) : (\mathbb{C}^2, a) \rightarrow (\mathbb{C}^2, F(a))$  be a germ of a holomorphic mapping. Assume that  $F$  has a generalized cusp at a point  $a \in \mathbb{C}^2$ . Since the curve  $J(F) = 0$  is reduced near  $a$ , we have that the set  $\{\nabla f = 0\} \cap \{\nabla g = 0\}$  has only isolated points near  $a$ . For a generic linear mapping  $T \in GL(2)$ , if  $F' = (f', g') = T \circ F$  then  $\nabla f'$  does not vanish identically on any branch of  $\{J(F) = 0\}$  near  $a$ . We say that the cusp of  $F$  at  $a$  has an index  $\mu_a := \dim_{\mathbb{C}} \mathcal{O}_a / (J(F'), J_{1,1}(F')) - \dim_{\mathbb{C}} \mathcal{O}_a / (f'_x, f'_y)$ .

**Remark 7.4.** We show below that the index  $\mu_a$  is well-defined and finite. Moreover, it is easy to see that a simple cusp has index one.

**Remark 7.5.** Using the exact sequence 1.7 from [4] we see that

$$\mu_a = \dim_{\mathbb{C}} \mathcal{O}_a / (J(F), J_{1,1}(F), J_{1,2}(F)).$$

Hence our index coincides with the classical local number of cusps defined e.g. in [4].

We have (compare with [4], [5], [6]):

**Theorem 7.6.** Let  $X \subset \mathbb{C}^m$  be a smooth surface. Let  $F = (f, g) \in \Omega_X(d_1, d_2)$  and assume that  $F$  has a generalized cusp at  $a \in \mathbb{C}^2$ . If  $U_a \subset X$  is a sufficiently small ball around  $a$  then  $\mu_a$  is equal to the number of simple cusps in  $U_a$  of a generic mapping  $F' \in \Omega_X(d'_1, d'_2)$ , where  $d'_1 \geq d_1, d'_2 \geq d_2$ , (and  $d'_i > 1$ ) which is sufficiently close to  $F$  in the natural topology of  $\Omega_X(d'_1, d'_2)$ .

*Proof.* We can assume that  $X = \mathbb{C}^2$  and  $\nabla f$  does not vanish identically on any branch of  $\{J(F) = 0\}$  near  $a$ . In particular we have  $\dim \mathcal{O}_a / (f_x, f_y) = \dim \mathcal{O}_a / (J(F), f_x, f_y) < \infty$ .

Let  $F_i = (f_i, g_i) \in \Omega_2(d'_1, d'_2)$  be a sequence of generic mappings, which is convergent to  $F$ . Consider the mappings  $\Phi = (J(F), J_{1,1}(F))$ ,  $\Phi_i = (J(F_i), J_{1,1}(F_i))$ ,  $\Psi = (\nabla f)$  and  $\Psi_i = (\nabla f_i)$ . Thus  $\Phi_i \rightarrow \Phi$  and  $\Psi_i \rightarrow \Psi$ .

Since  $a$  is a cusp of  $F$  we have  $\Phi(a) = 0$ . Moreover  $d_a(\Phi) < \infty$ , where  $d_a(\Phi)$  denotes the local topological degree of  $\Phi$  at  $a$ . Indeed, if  $J_{1,1}(F) = 0$  on some branch  $B$  of the curve  $J(F) = 0$  then the rank of  $F|_B$  would be zero and by Sard theorem  $F$  has to contract  $B$ , which is a contradiction ( $F_a$  is proper). By the Rouché Theorem (see [2], p. 86), we have that for large  $i$  the mapping  $\Phi_i$  has exactly  $d_a(\Phi)$  zeroes in  $U_a$  and  $\Psi_i$  has exactly  $d_a(\Psi)$  zeroes in  $U_a$  (counted with multiplicities, if  $\Psi(a) \neq 0$  we put  $d_a(\Psi) = 0$ ). However, the mappings  $F_i$  are generic, in particular all zeroes of  $\Phi_i$  and  $\Psi_i$  are simple. Moreover the zeroes of  $\Phi_i$  which are not cusps of  $F_i$  are zeroes of  $\Psi_i$ . Hence  $\mu_a = d_a(\Phi) - d_a(\Psi)$  is indeed the number of simple cusps of  $F_i$  in  $U_a$ .  $\square$

**Corollary 7.7.** Let  $F \in \Omega_X(d_1, d_2)$  and  $d_i > 1$ . Assume that  $F$  has generalized cusps at points  $a_1, \dots, a_r$ . Then  $\sum_{i=1}^r \mu_{a_i} \leq c_X(d_1, d_2)$ . In particular the numbers of singular germs  $\{F_a, a \in X\}$  which are finitely determined and are not folds, is bounded by the number  $c_X(d_1, d_2)$ .

*Proof.* We prove only the last statement. Let  $F_a$  be a singular germ which is finitely determined. Then the curve  $J(F_a)$  is reduced. There are two possibilities:

- 1) the point  $F(a)$  is a non-singular point of  $\Delta(F)$ ,
- 2) the point  $F(a)$  is a singular point of  $\Delta(F)$ .

In the case 1) we have by [9] that  $F_a$  is equivalent to the germ  $(x, y) \rightarrow (x^k, y)$  and since  $J(F_a)$  is reduced we have  $k = 2$ , i.e.,  $F_a$  is a fold.

In the case 2)  $F_a$  is a generalized cusp. Hence the number of such germs is bounded by the number of generalized folds.  $\square$

**Remark 7.8.** Of course for  $X = \mathbb{C}^2$  or  $X = S$  the assumption  $d'_i > 1$  (or  $d_i > 1$ ) in Theorem 7.6 and Corollary 7.7 is not necessary.

## REFERENCES

- [1] J. M. Boardman, *Singularities of differentiable maps*, Publicationes Mathematicae de l'IHES, 33, (1967), 21–57.
- [2] E. M. Cirka, *Complex analytic sets*, Nauka, Moscow, (1985) (in Russian).
- [3] T. Fukuda, G. Ishikawa, *On the number of cusps of stable perturbations of a plane-to-plane singularity*, Tokyo J. Math. 10 (1987), no. 2, 375–384.
- [4] T. Gaffney, D.M.Q. Mond, *Cusps and double folds of germs of analytic maps  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$* , J. London Math. Soc. (2) 43 (1991), no. 1, 185–192.
- [5] T. Gaffney, D.M.Q. Mond, *Weighted homogeneous maps from the plane to the plane*, Math. Proc. Cambridge Philos. Soc. 109 (1991), no. 3, 451–470.
- [6] T. Gaffney, *Polar multiplicities and equisingularity of map germs*, Topology 32 (1993), no. 1, 185–223.
- [7] R. Gunning, H. Rossi, *Analytic functions of several variables*, Prentice Hall (1965).
- [8] M. Golubitsky, V. Guillemin, *Stable mappings and their singularities*, GTM, Springer-Verlag (1973).
- [9] Z. Jelonek, *On finite regular and holomorphic mappings*, Advances in Math., 306, (2017), 1377–1391.
- [10] Z. Jelonek, *On semi-equivalence of generically-finite polynomial mappings*, Math. Z., 283,(2016), 133–142.
- [11] I. Krzyżanowska, Z. Szafraniec, *On polynomial mappings from the plane to the plane*, J. Math. Soc. Japan Vol. 66, no 3 (2014), 805–818.
- [12] J.N. Mather, *On Thom-Boardman singularities*, Dynamical Systems Proceedings of a Symposium Held at the University of Bahia, Salvador, Brasil, July 26-August 14, 1971, (1973), 233–248.
- [13] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, Princeton University Press, (1968).
- [14] J. J. Nuno-Ballesteros, B. Orefice-Okamoto, J. N. Tomazella, *Equisingularity of map germs from a surface to the plane*, arXiv:1507.01483v3 [math.AG] 7 Jun 2016.
- [15] T. Ohmoto, *Singularities of maps and characteristic classes*, arXiv:1309.0661v3 [math.AG] 31 Jan 2014.
- [16] J. Rieger, *Families of maps from the plane to the plane*, J. London Math. Soc. 36 (1987), 351–369.
- [17] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) 62 (1955), 374–410.

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